MATH 3150 – HOMEWORK 2

Updated 9/16/13: Problem 6 has been added.

Problem 1. Define x_n inductively by $x_1 = \sqrt{2}$, $x_n = \sqrt{2 + x_{n-1}}$. This is shown in Example 1.2.10 to be increasing and bounded. Let $\lambda = \lim_{n \longrightarrow \infty} x_n$.
(a) Show that λ is a root of $\lambda^2 - \lambda - 2 = 0$.
(b) Find λ.

Problem 2. Let x_n be a monotone increasing sequence such that $x_{n+1} - x_n \leq 1/n$. Must x_n converge?

Problem 3. Show that $d = \inf(S)$ if and only if d is a lower bound for S and for any $\epsilon > 0$ there is an $x \in S$ such that $d \geq x - \epsilon$.

Problem 4. Let x_n be a monotone increasing sequence bounded above and consider the set $S = \{x_1, x_2, \ldots\}$. Show that x_n converges to $\sup(S)$. Make a similar statement for decreasing sequences.

Remark. This shows that the *least upper bound property* — that every nonempty set with an upper bound has a least upper bound — implies the *monotone sequence property* — that every monotone increasing bounded sequence bounded above converges. Combined with the reverse implication proved in class, it follows that the least upper bound property is an equivalent to completeness.

Problem 5. For nonempty sets $A, B \subset \mathbb{R}$, let $A + B = \left\{ x + y \;\middle|\; x \in A \text{ and } y \in B \right\}$. Show that $\sup(A+B) = \sup(A) + \sup(B)$.