MATH 3150 – HOMEWORK 7

Update 11/2: Typo fixed in problem 5.
Update 11/4: Hint added in problem 4.

Problem 1 (p. 172, #1)**.** Which of the following sets are connected? Which are compact?
(a) $\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq 1\}$
(b) $\{x \in \mathbb{R}^n \mid \|x\| \leq 10\}$
(c) $\{x \in \mathbb{R}^n \mid 1 \leq \|x\| \leq 2\}$
(d) $\mathbb{Z} = \{\text{integers in } \mathbb{R}\}$
(e) a finite set in $\mathbb{R}$
(f) $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ (Be careful with the case $n = 1$!)
(g) Boundary of the unit square in $\mathbb{R}^2$
(h) The boundary of a bounded set in $\mathbb{R}$
(i) The rationals in $[0, 1]$
(j) A closed set in $[0, 1]$

Problem 2 (p. 173, #9)**.** Determine (by proof or counterexample) the truth or falsity of the following statements:
(a) (A is compact in $\mathbb{R}^n$) $\implies$ ($\mathbb{R}^n \setminus A$ is connected).
(b) (A is connected in $\mathbb{R}^n$) $\implies$ ($\mathbb{R}^n \setminus A$ is connected).
(c) (A is connected in $\mathbb{R}^n$) $\implies$ (A is open or closed).
(d) ($A = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$) $\implies$ ($\mathbb{R}^n \setminus A$ is connected). (Be careful with the case $n = 1$!)

Problem 3 (p. 174, #21)**.**
(a) Prove that a set $A \subset (M, d)$ is connected if and only if $\emptyset$ and A are the only subsets of A that are open and closed relative to A. (A set $U \subset A$ is called *open relative to* A if $U = V \cap A$ for some open set $V \subset M$; 'closed relative to A' is defined similarly.)
(b) Prove that $\emptyset$ and $\mathbb{R}^n$ are the only subsets of $\mathbb{R}^n$ that are both open and closed.

Problem 4 (p. 176, #38)**.** Show that $A \subset (M, d)$ is not connected if and only if there exist two disjoint open sets U, V such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$ and $A \subset U \cup V$. (In other words, we can change the definition of separating open sets to include the stronger condition that U and V are disjoint: $U \cap V = \emptyset$, without affecting which sets are connected under the new definition.)

[Hint: show that, if A is separated (in the sense defined in class) by open sets U and V, then for every $x \in A \cap U$ there exists $\epsilon > 0$, possibly depending on x, such that $D(x, \epsilon) \cap D(y, \epsilon) = \emptyset$ for all $y \in A \cap V$. Likewise for every point $y \in A \cap V$.]

Problem 5 (p. 176, #39)**. The Cantor Set:** Let $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ be obtained from $[0, 1]$ by removing the open middle third. Repeat the process, obtaining

$$F_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

In general F_n is a union of closed intervals and F_{n+1} is obtained by removing the middle third of each of these intervals. Define the *Cantor Middle Thirds Set* by

$$C = \bigcap_{n=1}^{\infty} F_n.$$

Prove:

(a) C is compact.
(b) C has infinitely many points. [Hint: Look at the endpoints of F_n.]
(c) $\text{int}(C) = \emptyset$.
(d) C is closed with no isolated points (i.e. every point is an accumulation point).
(e) C is *totally disconnected*, meaning if $x, y \in C$ and $x \neq y$, then there exist separating open sets U and V for C with $x \in U$ and $y \in V$.

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