MATH 3150 – HOMEWORK 8

Problem 1 (p. 184, #1)**.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Which of the following sets are necessarily closed, open, compact, or connected?
(a) $\{x \in \mathbb{R} \mid f(x) = 0\}$.
(b) $\{x \in \mathbb{R} \mid f(x) > 1\}$.
(c) $\{f(x) \in \mathbb{R} \mid x \geq 0\}$.
(d) $\{f(x) \in \mathbb{R} \mid 0 \leq x \leq 1\}$.

Problem 2 (p. 184, #3)**.** Give an example of a continuous map $f : \mathbb{R} \longrightarrow \mathbb{R}$ and a closed subset $B \subset \mathbb{R}$ such that $f(B)$ is not closed. Is this possible if B is bounded as well?

Problem 3 (p. 231, #1)**.**
(a) Prove directly (i.e. with 'ε's and 'δ's) that the function $1/x^2$ is continuous on $(0, \infty)$.
(b) A constant function $f : A \longrightarrow \mathbb{R}^m$ is a function such that $f(x) = f(y)$ for all $x, y \in A$. Show that f is continuous.
(c) Is the function $f(y) = 1/(y^4 + y^2 + 1)$ continuous? Justify your answer.

Problem 4 (p. 232, #7)**.** Consider a compact set $B \subset \mathbb{R}^n$ and let $f : B \longrightarrow \mathbb{R}^m$ be continuous and one-to-one (injective). Then prove that $f^{-1} : f(B) \longrightarrow B$ is continuous. Show by counterexample that this may fail if B is not compact. (To find a counterexample, it is necessary to take $m > 1$.)

Problem 5 (p. 232, #9)**.** Prove the following "gluing lemma": *Let $f : [a, b] \longrightarrow \mathbb{R}^m$ and $g : [b, c] \longrightarrow \mathbb{R}^m$ be continuous, and such that $f(b) = g(b)$. Define $h : [a, c] \longrightarrow \mathbb{R}^m$ by $h = f$ on $[a, b]$ and $h = g$ on $[b, c]$. Then h is continuous.* Generalize this result to sets $A, B \subset (M, d)$ in a metric space, with functions $f : A \longrightarrow (N, \rho)$ and $g : B \longrightarrow (N, \rho)$.