MATH 3150 — HOMEWORK 2

Problem 1 (p. 97, #5)**.** Let x_n be a monotone increasing sequence bounded above and consider the set $S = \{x_1, x_2, \ldots\}$. Show that x_n converges to $\sup(S)$. Make a similar statement for decreasing sequences.

Remark. This shows that the *least upper bound property* — that every nonempty set with an upper bound has a least upper bound — implies the *monotone sequence property* — that every monotone increasing bounded sequence bounded above converges. Combined with the reverse implication proved in class, it follows that the least upper bound property is equivalent to completeness.

Problem 2 (p. 97, #7)**.** For nonempty sets $A, B \subset \mathbb{R}$, let $A+B = \left\{x + y \;\middle|\; x \in A \text{ and } y \in B\right\}$. Show that $\sup(A + B) = \sup(A) + \sup(B)$.

Problem 3 (p. 52, #4)**.**

(a) Let x_n be a Cauchy sequence. Suppose that for every $\varepsilon > 0$ there is some $n > 1/\varepsilon$ such that $|x_n| < \varepsilon$. Prove that $x_n \longrightarrow 0$.

(b) Show that the hypothesis that x_n be Cauchy in (a) is necessary, by coming up with an example of a sequence x_n which does not converge, but which has the other property: that for every $\varepsilon > 0$ there exists some $n > 1/\varepsilon$ such that $|x_n| < \varepsilon$.

Problem 4 (p. 99 #15)**.** Let x_n be a sequence in $\mathbb{R}$ such that $|x_n - x_{n+1}| \leq \frac{1}{2} |x_{n-1} - x_n|$. Show that x_n is a Cauchy sequence.

Problem 5. Prove that an Archimedean ordered field in which every Cauchy sequence converges is complete (i.e. has the monotone sequence property). Here are some suggested steps:

(a) Denote the field by $\mathbb{F}$, and suppose x_n is a monotone increasing sequence bounded above by some $M \in \mathbb{F}$.

(b) Proceeding by contradiction, suppose x_n is not Cauchy. Deduce the existence of a subsequence $y_k = x_{n_k}$ with the property that

$$y_k \geq y_{k-1} + \varepsilon, \ \forall\ k \tag{1}$$

for some fixed positive number $\varepsilon > 0$ which does not depend on k.

(c) Using the Archimedean property, argue that y_k cannot be bounded above by M, hence obtaining a contradiction.

(d) Conclude that x_n converges.