# MATH 3150 — HOMEWORK 3

**Problem 1** (p. 70, #1, #3, #4)**.** This problem concerns the vector space $C([0,1])$ of continuous, real-valued functions $f:[0,1]\longrightarrow\mathbb{R}$, equipped with the inner product and two different norms:

$$\langle f,g\rangle=\int_0^1 f(x)g(x)\,dx,\quad \|f\|_2=\sqrt{\langle f,f\rangle},\quad \|f\|_\infty=\sup\{|f(x)|:x\in[0,1]\}.$$

(a) For $f(x)=1$ and $g(x)=x$, find $d(f,g)$ for both the sup norm $\|\cdot\|_\infty$ and the $L^2$-norm $\|\cdot\|_2$.
(b) Verify the Cauchy-Schwarz inequality for $f(x)=1$ and $g(x)=x$.
(c) Verify the triangle inequality for $f(x)=x$ and $g(x)=x^2$ in both norms.

**Problem 2** (p. 108, #4)**.** Let $B\subset\mathbb{R}^n$ be any set. Define

$$C=\{x\in\mathbb{R}^n : d(x,y)<1 \text{ for some } y\in B\}.$$

Show that $C$ is open.

**Problem 3** (p. 108, #6)**.**
(a) In $\mathbb{R}^2$, show that

$$\|x\|\le\|x\|_1\le\sqrt{2}\,\|x\|$$

where $\|x\|_1=|x_1|+|x_2|$ is the taxicab norm, and $\|x\|=\sqrt{x_1^2+x_2^2}$ is the usual Euclidean norm.
(b) Use the results of the first part to show that $\mathbb{R}^2$ with the taxicab metric $d_1(x,y)=|x_1-y_1|+|x_2-y_2|$ has the same open sets as it does with the standard metric $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$. In other words, show that every set which is open with respect to $d$ is also open with respect to $d_1$ and vice versa.

**Problem 4** (p. 145, #12, #14)**.** Prove the following properties for subsets $A$ and $B$ of a metric space:
(a) $\mathrm{int}(\mathrm{int}(A))=\mathrm{int}(A)$.
(b) $\mathrm{int}(A\cup B)\supset\mathrm{int}(A)\cup\mathrm{int}(B)$.
(c) $\mathrm{int}(A\cap B)=\mathrm{int}(A)\cap\mathrm{int}(B)$.
(d) $\mathrm{cl}(\mathrm{cl}(A))=\mathrm{cl}(A)$.
(e) $\mathrm{cl}(A\cap B)\subset\mathrm{cl}(A)\cap\mathrm{cl}(B)$.
(f) $\mathrm{cl}(A\cup B)=\mathrm{cl}(A)\cup\mathrm{cl}(B)$.

**Problem 5** (p. 143, #1, #2)**.** Determine whether the following sets are open or closed, and each set find its interior, closure and boundary.
(a) $(1,2)$ in $\mathbb{R}^1=\mathbb{R}$
(b) $[2,3]$ in $\mathbb{R}$
(c) $\bigcap_{n=1}^\infty[-1,1/n)$ in $\mathbb{R}$
(d) $\mathbb{R}^n$ in $\mathbb{R}^n$
(e) $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$
(f) $\{r\in(0,1)\mid r \text{ is rational}\}$ in $\mathbb{R}$
(g) $\{(x,y)\in\mathbb{R}^2\mid 0<x\le1\}$ in $\mathbb{R}^2$
(h) $\{x\in\mathbb{R}^n\mid \|x\|=1\}$ in $\mathbb{R}^n$
(i) $\{x_k\in\mathbb{R}^n\}$ for a sequence $x_k$ in $\mathbb{R}^n$ with no repeated terms.