Math 3150 Fall 2015 HW1 Solutions

Problem 1. Prove $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all positive integers n.

Solution. The proof is by induction. The base case, $n = 1$ states that $3 = 4(1)^2 - 1$, which is true. Suppose then that

$$3 + \cdots + (8n - 5) = 4n^2 - n$$

and consider the sum $3 + \cdots + (8n - 5) + (8(n + 1) - 5)$. By the inductive hypothesis, we have

$$\begin{aligned} 3 + \cdots + (8n - 5) + (8(n + 1) - 5) &= (4n^2 - n) + (8(n + 1) - 5) \\ &= 4n^2 + 8n + 4 - n - 1 \\ &= 4(n + 1)^2 - (n + 1), \end{aligned}$$

which completes the inductive step. □

Problem 2. In an ordered field, show that the following identities hold:

(iv) $(-a)(-b) = ab$ for all a, b;
(v) $ac = bc$ and $c \neq 0$ implies $a = b$.

Solution.

(iv) By part (iii) of Theorem 3.1, we have $(-a)(-b) = -(a(-b))$, and by commutativity of multiplication and (iii) again, we have $a(-b) = -ab$, so that

$$(-a)(-b) = -(-ab),$$

the additive inverse of the element $-ab$. However, since $ab + (-ab) = 0$ and additive inverses are unique, we conclude that $(-a)(-b) = ab$.

(v) Suppose $ac = bc$ and $c \neq 0$. By axiom (M4), there exists an element c^{-1} such that $c\,c^{-1} = 1$. Multiplying both sides of $ac = bc$ on the right by c^{-1} and using associativity of multiplication (M1), we have

$$\begin{aligned} & ac = bc \\ \Longrightarrow\ & (ac)c^{-1} = (bc)c^{-1} \\ \Longrightarrow\ & a(cc^{-1}) = b(cc^{-1}) \\ \Longrightarrow\ & a \cdot 1 = b \cdot 1 \\ \Longrightarrow\ & a) = b. \quad \square \end{aligned}$$

Problem 3. In an ordered field, show that the following identities hold:

(v) $0 < 1$;
(vii) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Solution.

(v) By multiplicative identity (M3), $1 = 1 \cdot 1 = 1^2$. By part (iv) of this theorem, $0 \leq a^2$ for all a, so we conclude $0 \leq 1$. However, $0 \neq 1$ is a field axiom,[1] so $0 < 1$.

[1] The book does not include this as an axiom, in which case the one point set $\{0\}$ is a field with respect to $0+0 = 0\cdot 0 = 0$. In this 'field', $0 = 1$. Most mathematicians do not regard $\{0\}$ as a field, and exclude it by requiring $0 \neq 1$ as a 'nontriviality' axiom.

(vii) Suppose $0 < a < b$. Since $a \neq 0$ and $b \neq 0$, there exist a^{-1} and b^{-1} such that $aa^{-1} = bb^{-1} = 1$. By part (vii) of the Theorem, $a^{-1} > 0$ and $b^{-1} > 0$, and then by part (iii), $a^{-1}b^{-1} \geq 0$. Multiplying both sides of $a < b$ by $a^{-1}b^{-1}$, we have, by (O5),

$$\begin{aligned} aa^{-1}b^{-1} &\leq ba^{-1}b^{-1} \\ \implies 1 \cdot b^{-1} &\leq a^{-1}bb^{-1} \\ \implies b^{-1} &\leq a^{-1} \cdot 1 \\ \implies b^{-1} &\leq a^{-1}. \end{aligned}$$

Furthermore, $b^{-1} \neq a^{-1}$ since otherwise, we would have $a = b$ by reversing the procedure (multiplying $a^{-1} = b^{-1}$ by ab implies $a = b$). □

Problem 4. Let $a, b \in \mathbb{R}$. Show if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

Solution. By contradiction, suppose $a > b$. By (a corollary of) the archimedean principle, there exists a number, call it b_1, such that

$$a > b_1 > b.$$

This contradicts the hypothesis that $a \leq b_1$ for every $b_1 > b$. □

Problem 5. Let S and T be nonempty subsets of $\mathbb{R}$ with the property that $s \leq t$ for all $s \in S$ and $t \in T$.

(a) Observe that S is bounded above and T is bounded below.
(b) Prove $\sup S \leq \inf T$.
(c) Give an example of such sets S and T where $S \cap T$ is nonempty.
(d) Give an example of such sets where $\sup S = \inf T$ but $S \cap T$ is the empty set.

Solution.

(a) Since T is nonempty, there exists some $t \in T$ and this has the property that $t \geq s$ for all $s \in S$; thus t is an uppper bound for S. Likewise, T is bounded below by an element $s \in S$.
(b) Let $s_0 = \sup S$ and $t_0 = \inf T$, and suppose, by contradiction, that $s_0 > t_0$. Since s_0 is the least upper bound for S, t_0 cannot be an upper bound for S, so there exists some $s \in S$ such that $s > t_0$. Since t_0 is the greatest lower bound for T, s can't be a lower bound for T, so there exists some $t \in T$ such that $s > t$, which contradicts the hypothesis that $s \leq t$ for all $s \in S$, $t \in T$.
(c) $S = [0, 1]$, $T = [1, 2]$, $\sup S = \inf T = 1$, $S \cap T = \{1\}$.
(d) $S = [0, 1)$, $T = (1, 2]$, $\sup S = \inf T = 1$, $S \cap T = \emptyset$. □

Problem 6. Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

Solution. By (a corollary of) the archimedean property, since $a > 0$, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\tfrac{1}{n_1} < a < n_2.$$

Taking $n = \max\{n_1, n_2\}$, we have $\frac{1}{n} \leq \frac{1}{n_1}$ and $n \geq n_2$, so $\frac{1}{n} < a < n$. □

Problem 7. Prove the following:

(a) $\lim \frac{(-1)^n}{n} = 0$
(b) $\lim \frac{1}{n^{1/3}} = 0$
(c) $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$
(d) $\lim \frac{n+6}{n^2-6} = 0$

Solution.

(a) Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then for all $n \geq N$,

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

(b) Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > \varepsilon^{-3}$. Then for all $n \geq N$,

$$\left|\frac{1}{n^{1/3}} - 0\right| = \frac{1}{n^{1/3}} \leq \frac{1}{N^{1/3}} < \varepsilon.$$

(Technically speaking, we should justify why $n \geq N$ implies $n^{1/3} \geq N^{1/3}$. By contradiction, suppose $m < M$, where $m = n^{1/3}$ and $M = N^{1/3}$. Then $m^2 < mM < M^2$ by two applications of the axiom which says that multiplication by positive elements preserves order, and likewise $m^3 < mM^2 < M^3$, which contradicts $n \geq N$.)

(c) Given $\varepsilon > 0$, let $N \geq \varepsilon^{-1}$. Then for all $n \geq N$,

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| = \left|\frac{2n-1-2(n+\frac{2}{3})}{3n+2}\right| = \left|\frac{-\frac{7}{3}}{3n+2}\right| = \frac{7}{9n+6}$$

$$\leq \frac{7}{9n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

(d) Note that if $n > 5$, then we have $n + 6 < 2n$ and if $n \geq 4$ then $n^2 - 6 \geq \frac{1}{2}n^2$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ so that $N > \max\left\{\frac{4}{\varepsilon}, 6\right\}$. Then for all $n \geq N$,

$$\left|\frac{n+6}{n^2-6} - 0\right| = \frac{n+6}{n^2-6} \leq \frac{2n}{\frac{1}{2}n^2} = \frac{4}{n} \leq \frac{4}{N} < \varepsilon.$$

□