Math 3150 Fall 2015 HW3 Solutions

Problem 1. Show that $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$ for bounded sequences (s_n) and (t_n).

Solution. Fix n and observe that $s_k \le \sup\{s_k : k \ge n\}$ and $t_k \le \sup\{t_k : k \ge n\}$ for all $k \ge n$. Thus $\sup\{s_k : k \ge n\} + \sup\{t_k : k \ge n\}$ is an upper bound for the set $\{s_k + t_k : k \ge n\}$ and must be greater than or equal to the least upper bound $\sup\{s_k + t_k : k \ge n\}$. In more compact notation, we have

$$a_n \le b_n + c_n, \quad \text{where}$$
$$a_n = \sup\{s_k + t_k : k \ge n\}, \quad b_n = \sup\{s_k : k \ge n\}, \quad c_n = \sup\{t_k : k \ge n\}.$$

Since these inequalities hold for all n, it follows that $\lim a_n \le \lim b_n + \lim c_n$. (This is the result of a homework problem we did not do, so it is worth mentioning a proof: to prove $a_n \le b_n \; \forall \; n \implies a := \lim a_n \le b := \lim b_n$, suppose by contradiction that $a > b$. Choosing $\varepsilon > 0$ such that $a - \varepsilon > b + \varepsilon$ (for instance $\varepsilon = a - b/4$ will do), it follows that there exist N_1 and N_2 such that $a_n > b_n$ for $n \ge \max(N_1, N_2)$, a contradiction.)

The conclusion follows since $\lim a_n = \limsup(s_n + t_n)$, $\lim b_n = \limsup s_n$ and $\lim c_n = \limsup t_n$. □

Problem 2. Show that $\limsup(s_n t_n) \le (\limsup s_n)(\limsup t_n)$ for bounded sequences (s_n) and (t_n) of nonnegative numbers.

Solution. By assumption $0 \le s_n$ and $0 \le t_n$ for all n, which implies that $0 \le \sup\{s_k : k \ge n\}$ for all n, and similarly $0 \le \sup\{t_k : k \ge n\}$. Fix $n \in \mathbb{N}$, and note that, for all $k \ge n$,

$$s_k t_k \le s_k \sup\{t_k : k \ge n\} \le \sup\{s_k : k \ge n\} \sup\{t_k : k \ge n\},$$

where we have twice used the fact that multiplication by nonnegative numbers preserves order. Thus the right hand side is an upper bound for the set $\{s_k t_k : k \ge n\}$, and therefore

$$\sup\{s_k t_k : k \ge n\} \le \sup\{s_k : k \ge n\} \sup\{t_k : k \ge n\} \quad \forall \; n.$$

This inequality persists in the limit as $n \to \infty$ (as noted in the previous proof), so we conclude that

$$\limsup(s_n t_n) = \lim_n \sup\{s_k t_k : k \ge n\} \le \lim_n \sup\{s_k : k \ge n\} \lim_n \sup\{t_k : k \ge n\} = (\limsup s_n)(\limsup t_n).$$

□

Problem 3. Let B be the set of all bounded sequences $\mathbf{x} = (x_1, x_2, \ldots)$ in $\mathbb{R}$.

(a) Define $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$. Show that d is a metric on B.

(b) Does $d^*(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} |x_i - y_i|$ define a metric on B?

Solution.

(a) The numbers $|x_i - y_i|$ are all nonnegative, which implies that $d(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in B$. Furthermore, if $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_i - y_i|\} = 0$ then $|x_i - y_i| = 0$ for all i, meaning that $x_i = y_i$ for all i and hence $\mathbf{x} = \mathbf{y}$. The symmetry condition $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ follows from $|x_i - y_i| = |y_i - x_i|$. Finally, for the triangle inequality, suppose $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are bounded sequences. We have

$$|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$$

from the triangle inequality for $|\cdot|$ in $\mathbb{R}$. From this it follows that

$$\sup\{|x_i - y_i|\} \leq \sup\{|x_i - z_i| + |z_i - y_i|\} \leq \sup\{|x_i - z_i|\} + \sup\{|z_i - x_i|\}, \qquad (1)$$

since $|x_i - z_i| + |z_i - y_i| \leq \sup\{|x_i - z_i|\} + \sup\{|z_i - y_i|\}$ for all i, The equation (1) is precisely the triangle inequality $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$. We conclude that d is a metric on B.

(b) The sequences are only supposed to be bounded, so the series $\sum_{i=1}^{\infty} |x_i - y_i|$ need not converge. For instance if $\mathbf{x} = (1, 1, 1, \ldots)$ and $\mathbf{y} = (0, 0, 0, \ldots)$, then $d^*(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} 1$ does not converge. Thus d^* is not defined on all pairs, and cannot be a metric. (If we limit ourselves to the set B^* of sequences $\mathbf{x} = (x_i)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$, then d^* is a metric on B^*.)

□

Problem 4. Let E be a subset of a metric space (S, d). Then

(a) E is closed if and only if $E = E^-$.

(b) E is closed if and only if it contains the limit of every convergent sequence of points in E.

(c) An element is in E^- if and only if it is the limit of a convergent sequence of points in E.

(d) Denoting the boundary of E by ∂E, we have $\partial E = E^- \cap (S \setminus E)^-$.

Proof.

(a) Suppose E is closed. Then E is the smallest closed set containing E, so $E = E^- = \bigcap\{C \supset E : C \text{ closed}\}$. Conversely, if $E = E^-$ then E is a union of closed sets, which is therefore closed.

(c) For this it is convenient to make use of the following Lemma, proved in class:

Lemma. *$x \in E^-$ if and only if for every $r > 0$ in $\mathbb{R}$, the open ball $B(x, r) = \{y \in S : d(x, y) < r\}$ contains some point of E.*

Suppose first that $x \in E^-$. Then for each $k \in \mathbb{N}$, we invoke the Lemma with $r = \frac{1}{k}$, and obtain a point $x_k \in E$. Together these form a sequence (x_k) with the property that $d(x_k, x) < \frac{1}{k}$, which implies $x_k \to x$.

Conversely, suppose x is the limit of a sequence (x_k) of points in E. Then given any $r > 0$, setting $\varepsilon = r$ in the definition of the limit gives an $N \in \mathbb{N}$ such that $d(x_N, x) < \varepsilon = r$. Since x_N is in E, this satisfies the hypothesis of the Lemma, so we conclude $x \in E^-$.

(b) This follows from (a) and (c). In more detail, if E is closed, then $E = E^-$ by part (a), and then E must contain the limit of every convergent sequence of points in E by the characterization of E^- in part (b).

Conversely, suppose E contains the limit of every convergent sequence of points in E. Such limits are precisely the points $x \in E^-$, so this means $E^- \subseteq E$. The inclusion $E \subseteq E^-$ always holds, so $E = E^-$ and then E is closed by part (a).

(b) By definition $\partial E = E^- \setminus E^\circ = E^- \cap (S \setminus E^\circ)$, so it suffices to show that $S \setminus E^\circ = (S \setminus E)^-$. One characterization of the interior is $E^\circ = \bigcup\{O \text{ open} : O \subseteq E\}$, so

$$S \setminus E^\circ = S \setminus \left(\bigcup\{O \text{ open} : O \subseteq E\}\right) = \bigcap\{S \setminus O : O \text{ open},\ O \subseteq E\}$$

since the complement of a union is the intersection of the complements. For each O, let $C = S \setminus O$. Then C is closed, and $O \subseteq E$ implies $C \supseteq (S \setminus E)$. Conversely, if C is a closed set containing $S \setminus E$, then $C = S \setminus O$, where O is open and contained in E. Thus

$$S \setminus E^\circ = \bigcap\{C : C \text{ closed},\ C \supseteq (S \setminus E)\} = (S \setminus E)^-. \qquad \square$$

Problem 5. Let (S, d) be any metric space.

(a) Show that a closed subset E of a compact set F is compact.

(b) Show that a finite union of compact sets is compact.

Solution.

(a) There are two natural proofs of this, using the two main characterizations of compact sets in terms of sequences and open covers, respectively.

Using open covers: Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an arbitrary open cover of E. Then $\mathcal{U} \cup \{S \setminus E\}$ is an open cover of F, since $S \setminus E$ is an open set, and any point in F is either in E, in which case is lies in some U_α, or it is in the complement of E, in which case it lies in $S \setminus E$. The cover has a finite subcover since F is compact. But since E is contained in F, this finite subcover is also a cover of E, and throwing out the set $S \setminus E$ if necessary, we obtain a finite subcover of $\mathcal{U}$ which covers E.

Using sequences: Let (s_n) be a sequence in E. Since $E \subset F$, (s_n) is also a sequence in F. Since F is compact, there exists a subsequence (s_{n_k}) such that $s_{n_k} \to s \in F$. Since E is closed, the limit, s, lies in E. We have produced a subsequence converging to a limit in E, and since (s_n) was arbitrary, we conclude that E is compact.

(b) Again we can give two proofs:

Using open covers: Let $\mathcal{U}$ be an open cover of $E_1 \cup \cdots \cup E_N$, where the E_i are compact. In particular $\mathcal{U}$ is an open cover of each E_i, $i = 1, \ldots, N$. Then for each i there is a finite open subcover cover: $E_i \subset U_{\alpha_{i,1}} \cup \cdots \cup U_{\alpha_{i,K_i}}$. Then

$$\left\{U_{\alpha_{i,n}} : 1 \le i \le N,\ 1 \le n \le K_i\right\}$$

is a finite subcover of $\mathcal{U}$ which covers $E_1 \cup \cdots \cup E_N$.

Using sequences: Suppose (s_n) is a sequence in $E_1 \cup \cdots \cup E_N$. There is some i such that infinitely many of the s_n lie in E_i; these form a subsequence of (s_n). Since E_i is compact, this has a further subsequence which converges in E_i. This subsubsequence is a subsequence of the original sequence which converges in $E_1 \cup \cdots \cup E_N$, and since (s_n) was arbitrary, we conclude that $E_1 \cup \cdots \cup E_N$ is compact.

$\square$

Problem 6. Determine which of the following series converge and justify your answers.

(a) $\sum_{n=2}^{\infty} \frac{1}{(n+(-1)^n)^2}$

(b) $\sum(\sqrt{n+1} - \sqrt{n})$

(c) $\sum \frac{n!}{n^n}$

Solution.

(a) By comparison,

$$\frac{1}{(n+(-1)^n)^2} \leq \frac{1}{(n-1)^2}$$

and $\sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the sequence converges.

(b) Multiplying and dividing by $\sqrt{n+1} - \sqrt{n}$, we have

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

By comparison,

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \frac{1}{2\sqrt{n+1}},$$

and $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n+1}} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n^p}$, $p = \frac{1}{2} \leq 1$ diverges, so the original series diverges.

(c) By the ratio test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \left(\frac{n}{n+1} \right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \to e^{-1} < 1,$$

so the series converges.

□

Problem 7.

(a) Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

(b) Observe that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ converges.

(c) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

Solution.

(a) The harmonic series $a_n = \frac{1}{n}$ is an example, since $\sum \frac{1}{n}$ diverges but $\sum \frac{1}{n^2}$ converges.

(b) Suppose $a_n \geq 0$ for all n and $\sum a_n$ converges. Then $a_n \to 0$ as a series, so in particular there exists some N such that $a_n \leq 1$ for all $n \geq N$. For such n, $a_n^2 \leq a_n$, so by comparison, the series $\sum_{n=M}^{\infty} a_n^2$ converges. The full series $\sum a_n^2$ differs from this by the finite sum $a_1^2 + \cdots + a_{N-1}^2$, so the full series $\sum a_n^2$ converges as well.

(c) One example would be the series $\sum a_n$ where $a_n = \frac{(-1)^n}{\sqrt{n}}$. By the alternating series test, this converges since the sequence $\frac{1}{\sqrt{n}} \to 0$. On the other hand, $a_n^2 = \frac{1}{n}$, giving the harmonic series which diverges.

□