Math 3150 Fall 2015 HW4 Solutions

Problem 1. Prove each of the following is continuous at x_0 by the ε–δ property.

(a) $f(x) = x^2$, $x_0 = 2$.

(b) $f(x) = \sqrt{x}$, $x_0 = 0$.

(c) $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$, $x_0 = 0$.

(d) $g(x) = x^3$, x_0 arbitrary.

Solution.

(a) Given $\varepsilon > 0$, set $\delta = \min(1, \varepsilon/5)$. Then for all x such that $|x - 2| \leq \delta$, we have

$$|f(x) - f(x_0)| = \left|x^2 - 4\right| = |x + 2|\,|x - 2| < 5\,|x - 2| < 5\frac{\varepsilon}{5} = \varepsilon.$$

(b) Note that the domain of f is $[0, \infty)$. Given $\varepsilon > 0$, set $\delta = \varepsilon^2$. Then for all $x \in [0, \infty)$ such that $|x - 0| = |x| < \delta$,

$$\left|\sqrt{x} - 0\right| = \sqrt{x} < \sqrt{\delta} = \varepsilon.$$

(c) Given $\varepsilon > 0$ let $\delta = \varepsilon$. Then $|x - 0| < \delta$ implies

$$|f(x) - 0| = \left|x \sin \tfrac{1}{x}\right| \leq |x| < \delta = \varepsilon.$$

(d) Given $\varepsilon > 0$, let $\delta = \min(1, \varepsilon/(3\,|x_0|^2 + 3\,|x_0| + 1))$. Then $|x - x_0| \leq \delta$ implies

$$\begin{aligned}
\left|x^3 - x_0^3\right| &= |x - x_0|\,\left|x^2 + x x_0 + x_0^2\right| \\
&\leq |x - x_0| \left(|x|^2 + |x|\,|x_0| + |x_0|^2\right) \\
&< |x - x_0| \left((|x_0| + 1)^2 + (|x_0| + 1)\,|x_0| + |x_0|^2\right) \\
&= |x - x_0| \left(3\,|x_0|^2 + 3\,|x_0| + 1\right) \\
&< \delta \left(3\,|x_0|^2 + 3\,|x_0| + 1\right) \leq \varepsilon.
\end{aligned}$$

□

Problem 2. For each nonzero rational number x, write x as $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ with no common factors and $q > 0$, and then define $f(x) = \frac{1}{q}$. Also define $f(0) = 1$ and $f(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Show that f is continuous at each point of $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous at each point of $\mathbb{Q}$.

Solution. First let x_0 be irrational; we show f is continuous at x_0. For each $q \in \mathbb{N}$, since x is not in the set $\left\{\frac{p}{q} : p \in \mathbb{Z}\right\}$, there exists a $\delta_q > 0$ such that $|x_0 - y| \geq \delta_q$ for all $y \in \left\{\frac{p}{q} : p \in \mathbb{Z}\right\}$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, and let $\delta = \min(\delta_1, \ldots, \delta_N)$. Then if $|x - x_0| < \delta$, it necessarily follows that x is either irrational or of the rational form $\frac{p}{q}$ for some $q > N$. Thus if $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| = |f(x)| < \frac{1}{N} < \varepsilon.$$

Now let x_0 be rational. To show f is discontinuous at x_0, it suffices to produce a sequence x_n such that $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$. Write $x_0 = \frac{p}{q}$ in reduced form. One way to do this is to let $x_n = \frac{np+1}{nq}$. Then $f(x_n) = \frac{1}{nq} \to 0$, while $f(x_0) = \frac{1}{q} \neq 0$. □

Problem 3. Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exist x, y in $[0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$.

Solution. Define the continuous function $g(x) = f(x + 1) - f(x)$ on $[0, 1]$. Then a pair x, y such that $|y - x| = 1$ and $f(x) = f(y)$ is equivalent to a number x such that $g(x) = 0$.

We examine several cases. First, if $g(0) = 0$ or $g(1) = 0$, then we already have a solution, so we can suppose without loss of generality that $g(0) \neq 0$ and $g(1) \neq 0$.

Note that $g(0) < 0$ implies $f(1) < f(0)$, and then since $f(2) = f(0)$, it follows that $f(2) > f(1)$ which is equivalent to $g(1) > 0$. Similarly, $g(0) > 0$ implies $g(1) < 0$. In either case, the interval $[g(0), g(1)]$ or $[g(1), g(0)]$ contains 0, so by the intermediate value theorem there exists $x \in (0, 1)$ such that $g(x) = 0$. □

Problem 4.

(a) Let $F(x) = \sqrt{x}$ for $x \geq 0$. Show f' is unbounded on $(0, 1]$ but f is nevertheless uniformly continuous on $(0, 1]$.

(b) Show f is uniformly continuous on $[1, \infty)$.

Solution.

(a) The derivative is $f'(x) = \frac{1}{2\sqrt{x}}$, which tends to $+\infty$ as $x \to 0$, so f' is unbounded on $(0, 1]$.

On the other hand, f is defined on the compact interval $[0, 1]$, so if f is just continuous (in the ordinary sense) on $[0, 1]$, then it is automatically uniformly continuous there. It suffices, therefore, to show that $f : [0, 1] \longrightarrow \mathbb{R}$ is continuous.

First, suppose $x_0 > 0$ and $\varepsilon > 0$ are given. Let $\delta = \min\left\{\frac{3x_0}{4}, \frac{3\sqrt{x_0}\varepsilon}{2}\right\}$. Then if $|x - x_0| < \delta$, it follows that $x > \frac{x_0}{4}$ and thus $\sqrt{x} > \frac{\sqrt{x_0}}{2}$. Then

$$\left|\sqrt{x} - \sqrt{x_0}\right| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} < \frac{|x - x_0|}{\frac{3}{2}\sqrt{x_0}} < \frac{\delta}{\frac{3}{2}\sqrt{x_0}} \leq \varepsilon.$$

For $x_0 = 0$, given $\varepsilon > 0$, we let $\delta = \varepsilon^2$. Then if $|x - 0| = x < \delta$, we have

$$\left|\sqrt{x} - 0\right| = \sqrt{x} < \varepsilon.$$

Thus f is continuous on $[0, 1]$, and hence uniformly continuous since $[0, 1]$ is compact. The restriction to any smaller subinterval, such as $(0, 1]$ is also uniformly continuous.

Note that this does not contradict the result that f' bounded implies f uniformly continuous, since this result is not an if and only if statement.

(b) On the interval $[1,\infty)$, f is again differentiable with $f'(x) = \frac{1}{2\sqrt{x}}$. However on this interval, $|f'(x)| \leq \frac{1}{2\sqrt{1}} = \frac{1}{2}$, so f' is bounded. We conclude that f is uniformly continuous on $[1,\infty)$.

Alternatively, it is possible to prove directly that f is uniformly continuous on $[0,\infty)$, as several of you did. Here is a nice proof: Given $\varepsilon > 0$ let $\delta = \varepsilon^2$. Then for any $x, y \in [0,\infty)$ such that $|x-y| \leq \delta$, either

1) $\left|\sqrt{x}+\sqrt{y}\right| = \sqrt{x}+\sqrt{y} < \varepsilon$, in which case

$$\left|\sqrt{x}-\sqrt{y}\right| \leq \sqrt{x}+\sqrt{y} < \varepsilon$$

by the triangle inequality, or

2) $\left|\sqrt{x}+\sqrt{y}\right| \geq \varepsilon$, in which case

$$\left|\sqrt{x}-\sqrt{y}\right| = \frac{|x-y|}{\left|\sqrt{x}+\sqrt{y}\right|} < \frac{\delta}{\varepsilon} = \varepsilon,$$

where we multiply and divide by $(\sqrt{x}+\sqrt{y})/(\sqrt{x}+\sqrt{y})$ inside the absolute value in the first step. □

Problem 5. Let $f(x) = x\sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$.

(a) Observe that f is continuous on $\mathbb{R}$.

(b) Why is f uniformly continuous on any bounded subset of $\mathbb{R}$?

(c) Is f uniformly continuous on $\mathbb{R}$?

Solution.

(a) For $x \neq 0$, $f(x)$ is the multiplication and composition of continuous functions $x \longmapsto x$, $x \longmapsto \sin x$ and $x \longmapsto \frac{1}{x}$, so is continuous. At $x = 0$ we proved continuity directly in class: given $\varepsilon > 0$ let $\delta = \varepsilon$. Then for $|x| \leq \delta$,

$$|f(x) - f(0)| = \left|x \sin\tfrac{1}{x}\right| \leq |x| < \varepsilon.$$

(b) Let $A \subset \mathbb{R}$ be a bounded subset, meaning $A \subset [-R,R]$ for some $R > 0$. Since $[-R,R]$ is compact, f is uniformly continuous on $[-R,R]$, and therefore on any subset thereof.

(c) In fact, f is uniformly continuous on all of $\mathbb{R}$. Note that f is differentiable on the intervals $(-\infty,-1]$ and $[1,\infty)$, with derivative $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$. For x in either of these intervals, we have

$$\left|f'(x)\right| \leq \left|\sin\left(\tfrac{1}{x}\right)\right| + \left|\tfrac{1}{x}\cos\left(\tfrac{1}{x}\right)\right| \leq 2.$$

It follows that f is uniformly continuous on $(-\infty,-1]$ and $[1,\infty)$. We noted above that f is uniformly continuous on $[-1,1]$. It is then a general fact that if a function f is separately uniformly continuous on a pair of intervals A and B meeting at a single point $A \cap B = \{p\}$, then it is uniformly continuous on $A \cup B$.

To prove this claim, let $\varepsilon > 0$ be given. There exist $\delta_A, \delta_B > 0$ such that if $x, x' \in A$ with $|x - x'| < \delta_A$, or if $x, x' \in B$ with $|x - x'| < \delta_B$, then $|f(x) - f(x')| < \varepsilon/2$. Then set $\delta = \min(\delta_A, \delta_B)$. If $x, x' \in A \cup B$ with $|x - x'| < \delta$, then

$$\left|f(x) - f(x')\right| < \varepsilon. \tag{1}$$

Indeed, if $x, x' \in A$ or $x, x' \in B$, then (1) follows from the above. In the case that $x \in A$ and $x' \in B$, say, it follows that $|x - p| < \delta_A$ and $|x' - p| < \delta_B$, where $p = A \cap B$ is the common endpoint. Then (1) follows from

$$\left|f(x) - f(x')\right| \leq |f(x) - f(p)| + \left|f(p) - f(x')\right| < \varepsilon/2 + \varepsilon/2.$$

$\square$

Problem 6. For metric spaces (S_1, d_1), (S_2, d_2) and (S_3, d_3), prove that if $f : S_1 \longrightarrow S_2$ and $g : S_2 \longrightarrow S_3$ are continuous, then $g \circ f : S_1 \longrightarrow S_3$ is continuous.

Solution. Let $U \subset S_3$ be an arbitrary open set. Then $g^{-1}(U) \subset S_2$ is open since g is continuous, and $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \subset S_1$ is open by continuity of f. Thus the inverse image of any open set in S_3 with respect to $g \circ f$ is open, and it follows that $g \circ f$ is continuous. $\square$

Problem 7. Show $C = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\right\}$ is a connected subset of $\mathbb{R}^2$.

Solution. We know that C is equivalent to the set of points $(\cos\theta, \sin\theta) \in \mathbb{R}^2$ such that $\theta \in [0, 2\pi)$. As several students pointed out, the map $f : [0, 2\pi) \longrightarrow \mathbb{R}^2$, $f(\theta) = (\cos\theta, \sin\theta)$ is continuous, and the domain is connected, hence the image $C = f([0, 2\pi))$ is connected.

Alternatively, we can proceed as follows. We first show that C is *path-connected*, meaning that for any pair of points $\vec{x}, \vec{y} \in C$, there is a continuous function (i.e., path) $\gamma : [a, b] \subset \mathbb{R} \longrightarrow C$ such that $\gamma(a) = \vec{x}$ and $\gamma(b) = \vec{y}$.

Indeed, $\vec{x} = (\cos\theta_0, \sin\theta_0)$ and $\vec{y} = (\cos\theta_1, \sin\theta_1)$ for some $\theta_0, \theta_1 \in [0, 2\pi)$, and then

$$\gamma(t) = \big(\cos((1 - t)\theta_0 + t\theta_1), \sin((1 - t)\theta_0 + t\theta_1)\big), 0 \leq t \leq 1$$

is a continuous path in C with $\gamma(0) = \vec{x}$ and $\gamma(1) = \vec{y}$.

Now we show that any path connected set is connected. Suppose, by contradiction, that C was disconnected, by open sets $U, V \subset \mathbb{R}^2$, say. Let $\vec{x} \in C \cap U$ and $\vec{y} \in C \cap V$ (which exist as these sets are nonemtpy), and let $\gamma : [0, 1] \longrightarrow C$ be a continuous path from $\vec{x}$ to $\vec{y}$. Then $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are open sets in $\mathbb{R}$ disconnecting $[0, 1]$, which is a contradiction since $[0, 1]$ is an interval, and hence is connected. $\square$