Math 3150 Fall 2015 HW5 Solutions

Problem 1. For each of the following series, find the radius of convergence and the exact interval of convergence.

(a) $\sum \sqrt{n} x^n$

(b) $\sum n^{-\sqrt{n}} x^n$

(c) $\sum x^{n!}$

(d) $\sum \frac{3^n}{\sqrt{n}} x^{2n+1}$

Solution.

(a) Here $a_n = \sqrt{n}$ and we may apply the ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \sqrt{\frac{n+1}{n}} \to 1$, which implies that $\lim |a_n|^{1/n} \to 1$ also. Hence the radius of convergence is 1. The series $\sum \sqrt{n}$ and $\sum \sqrt{n}(-1)^n$ both diverge (the associated sequences don't have limit 0), so the power series converges on $(-1, 1)$.

(b) We apply the root test directly: $a_n^{1/n} = 1/(n^{1/\sqrt{n}})$, and an argument similar to the proof of Theorem 9.7.(c) shows that $n^{1/\sqrt{n}} \to 1$, so $\lim a_n^{1/n} = 1$ and the radius of convergence is also 1. At $x = -1$, the series $\sum n^{-\sqrt{n}}(-1)^n$ converges by the alternating series test, since $n^{-\sqrt{n}} \to 0$. At $x = +1$, the series $\sum n^{-\sqrt{n}}$ converges by comparison to $\sum \frac{1}{n^2}$, since $n^{\sqrt{n}} > n^2$ for sufficiently large n. The power series converges on $[-1, 1]$.

(c) We may view $\sum x^{n!}$ as the power series $\sum a_k x^k$, where $a_k = 0$ unless $k = n!$, in which case $a_k = 1$. Then $\limsup |a_k|^{1/k} = 1$, so the radius of convergence is 1. If $|x| = 1$, then $|x|^{n!} \not\to 0$, and the series diverges. Thus the interval of convergence is $(-1, 1)$.

(d) Relabeling the series as $\sum_k a_k x^k$, where

$$a_k = \begin{cases} \frac{3^{(k-1)/2}}{\sqrt{\frac{k-1}{2}}} & k \text{ odd} \\ 0 & k \text{ even} \end{cases},$$

we can use the root test to compute

$$\begin{aligned} \limsup |a_k|^{1/k} &= \lim |a_{2n+1}|^{1/(2n+1)} \\ &= \lim \left(\frac{3^n}{\sqrt{n}}\right)^{1/(2n+1)} \\ &= \lim \frac{3^{n/(2n+1)}}{n^{1/(4n+2)}} \\ &= \lim 3^{1/2}(3n)^{-1/(4n+2)} = 3^{1/2}, \end{aligned}$$

where we use a similar proof to that of Theorem 9.7.(c) to show $(3n)^{1/(4n+2)} \to 1$. Thus the radius of convergence is $R = 1/\sqrt{3}$.

At $x = R$, the series

$$\sum_n \frac{3^n}{\sqrt{n}} \left(\frac{1}{\sqrt{3}}\right)^{2n+1} = \sum_n \frac{1}{\sqrt{3n}}$$

diverges since it is (up to a constant) of the form $\sum n^{-p}$ for $p \geq 1$. Likewise, at $x = -R$, the series

$$\sum_n \frac{3^n}{\sqrt{n}} \left(\frac{-1}{\sqrt{3}}\right)^{2n+1} = \sum_n \frac{-11}{\sqrt{3n}}$$

also diverges. Thus the series converges on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

□

Problem 2.

(a) Suppose $\sum a_n x^n$ has finite radius of convergence R and $a_n \geq 0$ for all n. Show that if the series converges at R, then it also converges at $-R$.

(b) Give an example of a power series whose interval of convergence is exactly $(-1, 1]$.

Solution.

(a) By assumption $\sum a_n R^n$ converges, which means in particular that the sequence $s_n = a_n R^n$ converges to 0. Then by the alternating series test, $\sum a_n (-R)^n = \sum s_n (-1)^n$ converges.

(b) The power series $\sum a_n x^n$ where $a_n = \frac{(-1)^n}{n}$ is an example.

□

Problem 3. For $x \in [0, \infty)$ let $f_n(x) = x/n$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

Solution.

(a) Fixing $x \in [0, \infty)$, we have $x/n \to 0$, so $f(x) = 0$ for all x, as the pointwise limit of (f_n).

(b) The convergence is uniform on $[0, 1]$. Indeed, given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that $N > 1/\varepsilon$. If $n \geq N$, then

$$\left|\tfrac{x}{n} - 0\right| = \tfrac{x}{n} \leq \tfrac{1}{N} < \varepsilon.$$

(c) The convergence is not uniform on $[0, \infty)$. To see this, recall that uniform convergence is equivalent to the statement that the sequence $b_n \to 0$, where

$$b_n = \sup\left\{|f_n(x) - f(x)| : x \in [0, \infty)\right\}.$$

But clearly for each n, $b_n = +\infty$, so this is not possible.

□

Problem 4. Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0,1]$.

(a) Does (f_n) converge pointwise on $[0,1]$? If so, find the limit function $f(x)$.

(b) Does (f_n) converge uniformly on $[0,1]$? Prove your assertion.

Solution.

(a) The sequence does converge uniformly: fixing $x \in [0,1]$, the sequence $(x + \frac{1}{n})^2$ converges to x^2, so $f(x) = x^2$ on $[0,1]$.

(b) The convergence is also uniform: indeed,

$$\left|\left(x - \frac{1}{n}\right)^2 - x^2\right| = \left|\frac{1}{n^2} - \frac{2x}{n}\right| \leq \left|\frac{1}{n^2} - \frac{2}{n}\right|$$

for all $x \in [0,1]$, and the latter sequence (which is independent of x), converges to 0.

□

Problem 5.

(a) Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1,1]$ to a continuous function.

(b) Does $\sum \frac{1}{n^2} x^n$ represent a continuous function on $[-1,1]$?

Solution.

(a) Consider the series $\sum g_k(x)$, where $g_k(x) = a_k x^k$. The Weierstrass M-test says that if we can find a sequence M_k such that $\sup_x |g_k(x)| \leq M_k$ and $\sum M_k$ converges, then $\sum g_k(x)$ converges uniformly. In this case we may take $M_k = |a_k|$, which converges by hypotheses. The partial sums $\sum_{k=1}^{n} a_k x^k$ are polynomials and therefore continuous on $[-1,1]$, and since the convergence is uniform the limit $\sum a_k x^k$ is continuous on $[-1,1]$ as well.

(b) Yes, by the above, $\sum \left|\frac{1}{n^2}\right| = \sum \frac{1}{n^2}$ converges, so $\sum \frac{1}{n^2} x^n$ is a continuous function on $[-1,1]$.

□