Math 3150 Fall 2015 HW6 Solutions

Problem 1.

(a) Observe that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$.

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

(c) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

Solution.

(a) We start with the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ on its interval of convergence, $(-1, 1)$. Differentiating this gives

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} nx^{n-1},$$

using differentiation termwise. Multiplying this by x gives part (a).

(b) The series in part (a) converges on $(-1, 1)$, so we may evaluate both sides at $x = \frac{1}{2}$, giving

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2.$$

(c) Likewise, we may evaluate at $x = \frac{1}{3}$ and $x = -\frac{1}{3}$ to obtain

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \frac{3}{4}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \frac{-1/3}{(1+1/3)^2} = -\frac{3}{16},$$

respectively.

□

Problem 2. Let $s(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $c(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$.

(a) Prove $s' = c$ and $c' = -s$.

(b) Prove $(s^2 + c^2)' = 0$.

(c) Prove $s^2 + c^2 = 1$.

Solution.

(a) Note that both series have infinite radius of convergence. We may differentiate termwise to get

$$s'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} = c(x),$$

and

$$c'(x) = \sum_{n=1}^{\infty} (-1)^n \frac{2nx^{2n-1}}{2n!} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} = -s(x).$$

(b) Using the sum and product rules for derivatives gives

$$(s^2(x) + c^2(x))' = (s^2(x))' + (c^2(x))' = 2s(x)c(x) - 2c(x)s(x) = 0.$$

(c) From part (b), $s^2 + c^2$ must be constant, and it suffices to evaluate it at any point. Choosing $x = 0$ gives

$$s(0) = \sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+1}}{(2n+1)!} = 0, \quad c(0) = \sum_{n=0}^{\infty} (-1)^n \frac{0^{2n}}{2n!} = 1,$$

so that $(s^2 + c^2)(0) = 1$, and hence $(s^2 + c^2)(x) = 1$ for all x.

□

Problem 3. Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(a) Show that f is differentiable at each $x \neq 0$ and calculate $f'(x)$.

(b) Use the definition to show that $f'(0) = 0$.

(c) Show that f' is not continuous at $x = 0$.

Solution.

(a) For $x \neq 0$, we may use the product and chain rules to compute

$$f'(x) = 2x \sin \tfrac{1}{x} - x^2 (\tfrac{1}{x^2}) \cos \tfrac{1}{x} = 2x \sin \tfrac{1}{x} - \cos \tfrac{1}{x}.$$

(b) At $x = 0$, we have

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = \left| \frac{x^2 \sin \frac{1}{x}}{x} \right| = \left| x \sin \tfrac{1}{x} \right| \leq |x| \to 0,$$

as $x \to 0$. Thus f is differentiable at $x = 0$ with $f'(0) = 0$.

(c) f' is not continuous at 0, since we may construct sequences (x_n) such that $\lim x_n = 0$ but $\lim f'(x_n) \neq f'(0)$. Indeed, let $x_n = \frac{1}{2\pi n}$. Then

$$f'(x_n) = \cos(2\pi n) = 1$$

for all n, so $\lim f'(x_n) = 1$ while $f'(0) = 0$.

□

Problem 4. Let $f(x) = x^2$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$.

(a) Prove f is continuous at $x = 0$.

(b) Prove f is discontinuous at all $x \neq 0$.

(c) Prove f is differentiable at $x = 0$

Solution.

(a) To show continuity at 0, we must show $\lim_{x\to 0} f(x) = f(0) = 0$. Given $\varepsilon > 0$, let $\delta = \sqrt{\varepsilon}$; then $0 < |x - 0| < \delta$ implies $|f(x) - 0| \le |x|^2 < \delta^2 = \varepsilon$.

(b) Let $x \neq 0$. If x is rational, set $\varepsilon = |x|^2$. Then for any $\delta > 0$, we may find an irrational y such that $|x - y| < \delta$, but $|f(x) - f(y)| = |x|^2 \ge \varepsilon$.

If x is irrational, set $\varepsilon = \frac{|x|}{2}$. Then for every $\delta > 0$ we may find a rational y such that $|x - y| < \delta$ and $|y|^2 \ge \varepsilon$, so that $|f(x) - f(y)| = |y|^2 \ge \varepsilon$.

□

Problem 5. Suppose f is differentiable on $\mathbb{R}$, $1 \le f'(x) \le 2$ for all $x \in \mathbb{R}$, and $f(0) = 0$. Prove that $x \le f(x) \le 2x$ for all $x \ge 0$.

Solution. The statement is obvious for $x = 0$, so select $x > 0$, and use the mean value theorem to write

$$f(x) = f(x) - f(0) = f'(y)(x - 0) = x\, f'(y)$$

for some $y \in [0, x]$. Using that $1 \le f'(y) \le 2$, we obtain

$$x \le f(x) = x f'(y) \le 2x.$$

□

Problem 6. Show that if f is integrable on $[a, b]$, then f is integrable on every interval $[c, d] \subseteq [a, b]$.

Solution. Let $\varepsilon > 0$ be given. We will show that there exists a partition P of $[c, d]$ such that $U(f, P) - L(f, P) < \varepsilon$. This same property holds for the interval $[a, b]$ by assumption; namely, there exists a partition Q of $[a, b]$ such that $U(f, Q) - L(f, Q) < \varepsilon$.

We may assume without loss of generality that $c, d \in Q$. (Adding these points to Q leads to a finer partition Q', for which $U(f, Q') \le U(f, Q)$ and $L(f, Q') \ge L(f, Q)$, so that $U(f, Q') - L(f, Q') < \varepsilon$ still holds.) In particular $P = Q \cap [c, d]$ is then a partition of $[c, d]$. We have

$$U(f, P) - L(f, P) = U(f, Q) - L(f, Q) - \sum_j \left(M(f, I_j) - m(f, I_j)\right) |I_j| ,$$

where the sum is over the intervals I_j which are in Q but not P, and $|I_j|$ denotes the length of the interval I_j. Since

$$M(f, I_j) - m(f, I_j) = \sup\{f(x) : x \in I_j\} - \inf\{f(x) : x \in I_j\} \ge 0,$$

it follows that $U(f, P) - L(f, P) \le U(f, Q) - L(f, Q) < \varepsilon$. □

Problem 7. Give an example of a function f on $[0, 1]$ that is *not* integrable but such that $|f|$ is integrable.

Solution. Let $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = -1$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Then f is not integrable since, in any nonzero interval I, $M(f, I) = 1$ while $m(f, I) = -1$, whence $U(f, P) = 1$ and $L(f, P) = -1$ for all partitions P of $[0, 1]$ and then $U(f) = 1 \neq L(f) = -1$.

On the other hand, $|f| = 1$ is easily seen to be integrable, since $U(|f|, P) = L(|f|, P) = 1$ for all partitions, so $\int_0^1 |f|(x)\, dx = U(f) = L(f) = 1$. □