POSITIVE NTH ROOTS

Here we prove, using the completeness property of $\mathbb{R}$, that positive nth roots exist. The proof is taken from Walter Rudin's book "Principles of mathematical analysis."

Theorem. *Let $x > 0$ be a positive real number. For every natural number $n \geq 1$, there exists a unique positive nth root of x, which is to say $y > 0 \in \mathbb{R}$ such that $y^n = x$.*

Proof. Given x we construct a set S whose supremum will be the desired element y. Let

$$S = \{t \in \mathbb{R} : t > 0,\ t^n \leq x\}.$$

First we claim that S is nonempty. To see this, consider $t = x/(x+1)$; note that $t < 1$ and $t < x$. Since $t < 1$, it follows that $t^{n-1} < 1$, so

$$t^n < t < x,$$

and thus $t \in S$. Next, to show that S has an upper bound, we consider the element $r = x+1$; note that $r > 1$ and $r > x$. Since $r > 1$, it follows that $r^{n-1} > 1$, and then

$$r^n \geq r > x,$$

so r is an upper bound: indeed, if $t \in S$, then $t^n \leq x < r^n$, so we must have $t \leq r$ (otherwise, if $t > r$, then we would have $t^n > r^n$).

By the completeness property of $\mathbb{R}$, S has a supremum. Let

$$y = \sup(S).$$

We now claim that $y^n = x$. To prove this, we will derive contradictions from the other two possibilities, that $y^n > x$ and $y^n < x$. Both steps are somewhat unintuitive, and use the following estimate

$$0 < a < b \implies b^n - a^n < (b-a)\, n\, b^{n-1},\ \forall\ n, \tag{1}$$

which follows by estimating $a^k < b^k$ in the formula

$$b^n - a^n = (b-a)\underbrace{(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1})}_{<n(b^{n-1})}.$$

Suppose $y^n < x$. Then we can choose a real number h such that

$$0 < h < \frac{x - y^n}{n(y+1)^{n-1}}, \quad h < 1.$$

Indeed, $(x - y^n) > 0$ by assumption and the denominator is positive, and we can always arrange $h < 1$ by making it smaller if necessary. Invoking the formula (1) with $a = y$ and $b = (y+h)$ gives

$$(y+h)^n - y^n < h\, n\, (y+h)^{n-1} < \frac{(x-y^n)n(y+h)^{n-1}}{n(y+1)^{n-1}} < \frac{(x-y^n)n(y+1)^{n-1}}{n(y+1)^{n-1}} = x - y^n.$$

Cancelling the y^n terms from both sides, we obtain $(y+h)^n < x$, so $y + h \in S$ and since $y < y + h$ this contradicts the fact that y is an upper bound for S.

Now suppose $y^n > x$. Then we define

$$k = \frac{y^n - x}{ny^{n-1}}$$

and note that

$$0 < k < \frac{y^n}{ny^{n-1}} = \frac{y}{n} < y.$$

Invoking the formula (1) again with $a = y - k$ (which we've just shown to be positive) and $b = y$, we obtain the estimate

$$y^n - (y-k)^n < k\,n\,y^{n-1} = \frac{(y^n - x)\,n\,y^{n-1}}{ny^{n-1}} = y^n - x.$$

Cancelling the y^n terms and multiplying by -1, it follows that $x < (y-k)^n$, i.e. that $y - k$ is an upper bound for S. Since $y - k < y$ this contradicts the property of the supremum that y is less than or equal to any other upper bound.

Since $y^n \not> x$ and $y^n \not< x$, it follows that $y^n = x$. □