MATH 3150 FINAL EXAM PRACTICE PROBLEMS – FALL 2014

Problem 1. Suppose (s_n) is a sequence in $\mathbb{R}$, and for each n, let $\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n)$.
(a) Show that, if (s_n) is convergent, then (σ_n) is convergent and $\lim \sigma_n = \lim s_n$.
(b) Find an example where (σ_n) converges but (s_n) does not.

Problem 2. Show that $f(x) = x^2$ is uniformly continuous on the open interval $(-1, 2)$.

Problem 3. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(a) Show that f is continuous, and uniformly continuous on $[-1, 1]$.
(b) Show that f is not differentiable at $x = 0$.

Problem 4. Let $f(x) = \int_0^{x^2} e^{\sqrt{t}}\, dt$ for $x \in [0, +\infty)$.
(a) Compute $f(0)$.
(b) Show that f is differentiable on $(0, +\infty)$ and compute $f'(x)$.

Problem 5. Define $f : [0, 1] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 2 & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2}. \end{cases}$$

Show that f is integrable and compute $\int_0^1 f(x)\, dx$.

Problem 6. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq C\, |x - y|^2, \quad \forall x, y \in \mathbb{R}$$

for some $C \geq 0$. Show that f must be constant. [Hint: show that it is differentiable first.]

Problem 7. Suppose $f : [0, +\infty) \longrightarrow \mathbb{R}$ is continuous and differentiable on $(0, +\infty)$, and suppose that

$$f(x) + x\, f'(x) \geq 0, \quad \forall x > 0.$$

Show that $f(x) \geq 0$ for all $x \geq 0$. [Hint: consider the function $g(x) = x f(x)$.]

Problem 8. Let $f_n : A \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sequence of functions (not necessarily continuous), converging uniformly to a function $f : A \subset \mathbb{R} \longrightarrow \mathbb{R}$. Show that, if each f_n is bounded, then f is bounded.