MATH 3150 FINAL EXAM PRACTICE PROBLEMS – FALL 2014

Problem 1. Suppose (s_n) is a sequence in $\mathbb{R}$, and for each n, let $\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n)$.
(a) Show that, if (s_n) is convergent, then (σ_n) is convergent and $\lim \sigma_n = \lim s_n$.
(b) Find an example where (σ_n) converges but (s_n) does not.

Solution.
(a) Suppose $s_n \to s$; we want to show that $\sigma_n \to s$ as well. The key estimate is

$$(1) \qquad |\sigma_n - s| = \left| \frac{s_1 + \cdots + s_n}{n} - s \right| = \left| \frac{s_1 + \cdots + s_n - ns}{n} \right| \leq \frac{|s_1 - s|}{n} + \cdots + \frac{|s_n - s|}{n},$$

where we use the triangle inequality (n times).

Let $\varepsilon > 0$ be given. Since $s_n \to s$, there exists $N_0 \in \mathbb{N}$ such that $n \geq N_0$ implies $|s_n - s| < \varepsilon/2$. Also, since (s_n) converges, it is bounded; in particular there exists $M > 0$ such that $|s_n - s| \leq M$ for all n. Using the latter to estimate the first N_0 terms in (1) and the former to estimate the rest, we have, for $n > N_0$,

$$|\sigma_n - s| < \frac{N_0 M}{n} + \frac{(n - N_0)\varepsilon}{2n} \leq \frac{N_0 M}{n} + \frac{n\varepsilon}{2n} = \frac{N_0 M}{n} + \frac{\varepsilon}{2}.$$

Now, there exists N_1 such that for $n \geq N_1$, $(N_0 M)/n < \varepsilon/2$, and then

$$n \geq \max\{N_0, N_1\} \implies |\sigma_n - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(b) The alternating sequence $(s_n) = (1, 0, 1, 0, 1, \ldots)$ does not converge, but

$$(\sigma_n) = (1, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{1}{2}, \ldots)$$

converges to $1/2$.

□

Problem 2. Show that $f(x) = x^2$ is uniformly continuous on the open interval $(-1, 2)$.

Solution. $f(x)$ is continuous, and restricted to a *closed* and bounded (hence compact) interval, say $[-1, 2]$, it is uniformly continuous. But then it is uniformly continuous on any subset thereof, such as $(-1, 2)$.

Alternatively, you can give a direct ε–δ proof. □

Problem 3. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(a) Show that f is continuous, and uniformly continuous on $[-1, 1]$.
(b) Show that f is not differentiable at $x = 0$.

Solution.

(a) f is continuous at any $x \neq 0$ since there it is the product of the continuous function x and the composition of the continuous functions $\sin(x)$ and $1/x$.

At $x = 0$ we must show that

$$\lim_{x\to 0} f(x) = f(0) = 0.$$

We will show that $\lim_{x\to 0} |f(x)| = 0$ which implies the above.

$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} |x| \left|\sin\left(\tfrac{1}{x}\right)\right| \leq \lim_{x\to 0} |x| = 0$$

since sine is bounded by 1 in absolute value.

Then f is uniformly continuous on any compact set, such as $[-1, 1]$ since this is true for any continuous function.

(b) The limit of the difference quotient

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} \sin\left(\tfrac{1}{x}\right)$$

does not exist. Therefore f is not differentiable at $x = 0$.

□

Problem 4. Let $f(x) = \int_0^{x^2} e^{\sqrt{t}}\, dt$ for $x \in [0, +\infty)$.

(a) Compute $f(0)$.

(b) Show that f is differentiable on $(0, +\infty)$ and compute $f'(x)$.

Solution.

(a) $f(0) = \int_0^0 e^{\sqrt{t}}\, dt = 0$ since the interval of integration has width 0.

(b) The integrand, $t \longmapsto e^{\sqrt{t}}$, is a continuous function on $[0, +\infty)$ and therefore by the fundamental theorem of calculus,

$$F(y) = \int_0^y e^{\sqrt{t}}\, dt$$

is differentiable with derivative $F'(y) = e^{\sqrt{y}}$. Now $f = F \circ g$ where $g(x) = x^2$, so using the chain rule,

$$f'(x) = e^{\sqrt{g(x)}} g'(x) = 2xe^x.$$

□

Problem 5. Define $f : [0, 1] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 2 & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2}. \end{cases}$$

Show that f is integrable and compute $\int_0^1 f(x)\, dx$.

Solution. It suffices to show that for any $\varepsilon > 0$, there exists a partition P of $[0, 1]$ such that

$$U(f, [0, 1], P) - L(f, [0, 1], P) < \varepsilon$$

since this implies that $\sup_P \{L(f, [0, 1], P)\}$ and $\inf_P \{U(f, [0, 1], P)\}$ — the lower and upper integrals, respectively — are equal.

For any partition $P = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$, the upper integral is

$$U(f, [0,1], P) = \sum_{i=1}^{N} \sup_{[x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = 2 \sum_{i=1}^{N} (x_i - x_{i-1}) = 2,$$

and the lower integral is

$$L(f, [0,1], P) = \sum_{i=1}^{N} \inf_{[x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = 2 \sum_{1/2 \notin [x_{i-1}, x_i]} (x_i - x_{i-1}) + 0 \cdot (x_j - x_{j-1})$$

where $[x_{j-1}, x_j]$ is the interval of P which contains the point $x = \frac{1}{2}$. Thus

$$L(f, [0,1], P) = 2(1 - (x_j - x_{j-1})).$$

Thus given any $\varepsilon > 0$, we may choose a partition such that the interval $[x_{j-1}, x_j]$ containing $x = 1/2$ has width $(x_j - x_{j-1}) < \varepsilon/2$. For such a partition P,

$$U(f, [0,1], P) - L(f, [0,1], P) < 2 - 2(1 - \varepsilon/2) = \varepsilon.$$

Thus f is integrable. Then

$$\int_0^1 f(x)\, dx = U(f) = 2$$

since the upper sums $U(f, [0,1], P)$ are all equal to 2 so their infimum is 2. □

Problem 6. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq C\, |x - y|^2, \quad \forall x, y \in \mathbb{R}$$

for some $C \geq 0$. Show that f must be constant. [Hint: show that it is differentiable first.]

Solution. By assumption, the limit

$$\lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim_{x \to y} C\, |x - y| = 0$$

which implies that

$$\lim_{x \to y} \frac{f(x) - f(y)}{x - y} = 0$$

and therefore f is differentiable at all $y \in \mathbb{R}$ with derivative $f'(y) = 0$. Since the derivative vanishes identically, f must be constant. □

Problem 7. Suppose $f : [0, +\infty) \longrightarrow \mathbb{R}$ is continuous and differentiable on $(0, +\infty)$, and suppose that

$$f(x) + x\, f'(x) \geq 0, \quad \forall x > 0.$$

Show that $f(x) \geq 0$ for all $x \geq 0$. [Hint: consider the function $g(x) = x f(x)$.]

Solution. If we define $g(x) = x\, f(x)$, then by the product rule,

$$g'(x) = f(x) + x\, f'(x) \geq 0, \quad \forall x > 0.$$

Thus g is increasing on $(0, +\infty)$. Furthermore,

$$g(0) = 0 \cdot f(0) = 0$$

so it follows that $g(x) \geq 0$ for $x > 0$.

Now, $f(x) = g(x)/x$ implies that $f(x) \geq 0$ for $x > 0$ since $1/x$ is positive there, and finally

$$\lim_{x \to 0} f(x) \geq 0$$

since the limit of a non-negative function is nonnegative. Thus $f(x) \geq 0$ for all $x \in [0, \infty)$. $\square$

Problem 8. Let $f_n : A \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sequence of functions (not necessarily continuous), converging uniformly to a function $f : A \subset \mathbb{R} \longrightarrow \mathbb{R}$. Show that, if each f_n is bounded, then f is bounded.

Solution. Since each f_n is bounded, there exists $M_n > 0$ such that $|f_n(x)| \leq M_n$ for all x. Then, since $f_n \to f$ uniformly, given $\varepsilon = 1$, there exists N such that

$$|f(x)| - |f_N(x)| \leq |f_N(x) - f(x)| < \varepsilon = 1, \quad \forall\, x \in A.$$

Combining this with $|f_N(x)| \leq M_N$ and rearranging, we have

$$|f(x)| < 1 + M_N, \quad \forall\, x \in A.$$

$\square$