Witten Deformation and Morse Theory

Jonier Amaral Antunes

April 25, 2016

Introduction

In 1982 Edward Witten published the paper “Supersymmetry and Morse theory”, [3], which was very influential on the research of mathematical physics, geometry and topology of the 1980’s. With the purpose of making important physical concepts present in the then emerging topic of supersymmetry accessible to a mathematical audience, Witten presented some models where supersymmetry was present and pointed out what type of properties would be pysically relevant and their connections with mathematics. While analyzing a model for sumpersymmetric (non-relativistic) quantum mechanics given by the algebra of differential forms on a manifold he gave an analytic argument to prove the Morse Inequalities.

Recall that $f \in C^\infty(M)$ is a Morse function on a n-dimensional compact oriented manifold M if all of its critical points are nondegenerate. That is, if for every $x \in M$ such that $df_x = 0$, the Hessian of f at x is non-singular. Since non-degenerate critical points are isolated, M being compact implies that f has only a finite number of critical points. The first important property of Morse functions is present in the following well-known result:

Theorem 0.1 (Morse lemma)**.** *For any critical point $x \in M$ of the Morse function f, there is an open neighborhood U_x of x not containing any other critical point and an oriented coordinate system $y = (y^1, \dots, y^n)$ on U_x such that*

$$f(y) = f(x) - \frac{1}{2}(y^1)^2 - \cdots - \frac{1}{2}(y^{n_f(x)})^2 + \frac{1}{2}(y^{n_f(x)+1})^2 + \cdots + \frac{1}{2}(y^n)^2$$

The integer $n_f(x)$ is called the Morse index of f at x and it does not depend on the coordinates. If we denote by m_i, the number of critical points of f with Morse index equal to i and by $\beta_i = \dim H^i_{dR}(M;\mathbb{R})$ the i-th Betti number of M, we can state the Morse inequalities:

Theorem 0.2. *(i)* Weak Morse inequalities*: For any* $0 \leq i \leq n$ *we have*

$$\beta_i \leq m_i$$

(ii) Strong Morse inequalities*: For any* $0 \leq i \leq n$ *we have*

$$\beta_i - \beta_{i-1} + \cdots + (-1)^i \beta_0 \leq m_i - m_{i-1} + \cdots + (-1)^i m_0$$

Moreover, for $i = n$*:*

$$\beta_n - \beta_{n-1} + \cdots + (-1)^n \beta_0 = m_n - m_{n-1} + \cdots + (-1)^n m_0$$

The Morse inequalities give us analytical tools to study the topology of a manifold through the behavior of the critical points of a Morse function. A proof of Theorem 0.2 using topological tools, as well as a proof of Morse lemma and further development of Morse theory can be found in [2]. In the present text we will follow the lines of Witten's argument thus obtaining a analytic proof for the inequalities.

Witten's idea consisted of deforming the exterior derivative by conjugation with a term e^{Tf} in way similar to the time evolution of operators representing observables in the Heisenberg picture of quantum mechanics. Under this time evolution, eigenfunctions of the kernel of the associated Laplace operator will get concentrated around the critical points of f, thus allowing a local analysis to be made.

Many works formalizing Witten's argument or extended its ideas to other areas followed the original paper, further strenghtening the connections between analysis, topology, geometry and mathematical physics. Here, we will follow the approach presented in [1], including most of its results and notation.

In Section 1 we present the prerequisites of differential geometry mainly with the purpose of fixing notation. We refer to [4] for details. Section 2 introduces Witten deformation and shows that the time evolution of the Laplace operator do not lose information about the Betti numbers. Sections 3 and 4 are more computationally heavy and present the local and global behavior, respectively, of the deformed Laplacian. These two sections are more technical and basically pave the way to obtaining Proposition 4.3, which is the main result that will allow us to prove the Morse inequalities in Section 5.

1 Preliminaries

From now on, let (M, g) be a n-dimensional compact oriented Riemannian manifold without boundary. The main result we want to prove, Theorem 0.2, does not depend on the metric g, thus some extra hypotheses will be included later about the behavior of g in the proximities of each critical point without affecting the generality of the proof.

Recall that with the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$, the algebra of differential forms $\Omega(M) = \bigoplus_{i=0}^{n} \Omega^i(M)$ can be seen as a cochain complex

$$(\Omega^\bullet(M), d) : 0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

called the de Rham complex, with corresponding cohomology given by the de Rham cohomology:

$$H^k_{\mathrm{dR}}(M; \mathbb{R}) := \frac{\ker d|_{\Omega^k(M)}}{\mathrm{Im}\ d|_{\Omega^{k-1}(M)}}$$

The Riemannian metric g induces an inner product on each $\Omega^k(M)$ via

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta \tag{1.1}$$

where $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ is the Hodge star operator. If we denote by $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ the differential operator given by

$$d^* = (-1)^{nk+n+1} \star d\star$$

it is easy to see that

$$\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle$$

That is, d and d^* are formally adjoint.

If we denote the de Rham- Hodge operator by $D = d + d^*$ then the Laplacian of the de Rham complex is the elliptic operator $D^2 = dd^* + d^*d$. In particular the Hodge decomposition theorem holds, giving us a decomposition

$$\Omega(M) = \ker D^2 \oplus \mathrm{Im}\ D^2$$

As a corollary of the decomposition theorem, we have the isomorphism

$$\ker D^2|_{\Omega^k(M)} \simeq H^k_{dR}(M; \mathbb{R}) \tag{1.2}$$

which allows us to obtain the Betti numbers via the kernel of D^2.

We can obtain convenient local expressions for these operators if we introduce the Clifford operators that give an action of TM on the exterior algebra bundle. If $v \in TM$, let us denote

$$c(v) := v^* \wedge - v\lrcorner \qquad \text{and} \qquad \hat{c}(v) := v^* \wedge + v\lrcorner$$

where $\lrcorner$ is the interior multiplication and $v^* \in T^*M$ is the dual of v under g. We will also denote $c(v^*) = c(v)$ and $\hat{c}(v^*) = \hat{c}(v)$. These maps satisfy the Clifford relations:

$$c(v)c(w) + c(w)c(v) = -2g(v,w)$$
$$\hat{c}(v)\hat{c}(w) + \hat{c}(w)\hat{c}(v) = 2g(v,w)$$
$$c(v)\hat{c}(w) + \hat{c}(w)c(v) = 0$$

Given a local orthonormal frame $e_1, \ldots, e_n$ for TM with corresponding dual basis $e^1, \ldots, e^n$ of T^*M and denoting by ∇ the connection induced on $\Omega(M)$ by the Levi-Civita connection, we have the following local expressions:

$$d = \sum_{i=1}^{n} e^i \wedge \nabla_{e_i} \tag{1.3}$$

$$d^* = -\sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i} \tag{1.4}$$

$$D = \sum_{i=1}^{n} c(e_i) \nabla_{e_i} \tag{1.5}$$

Equation (1.5), which is a direct consequence of (1.3) and (1.4), is presenting D as a Dirac operator.

2 Witten Deformation

Let $f \in C^\infty(M)$ be a fixed Morse function on M and for every $T \in \mathbb{R}$ define the deformation of the exterior diferential operator as:

$$d_{Tf} = e^{-Tf} d\, e^{Tf} \tag{2.1}$$

Since the algebra of differential forms is a module over $C^\infty(M)$ and multiplication by a function does not affect the grading, the deformation defined above can stilll be seen as an operator $d_{Tf} : \Omega^k(M) \to \Omega^{k+1}(M)$, for any k. Also, as an immediate consequence of the definition (2.1), we see that

$$d_{Tf}^2 = e^{-Tf} d^2 e^{Tf} = 0$$

Therefore, we have a deformation of the de Rham complex, given by the cochain complex

$$(\Omega^\bullet(M), d_{Tf}) : 0 \to \Omega^0(M) \xrightarrow{d_{Tf}} \Omega^1(M) \xrightarrow{d_{Tf}} \cdots \xrightarrow{d_{Tf}} \Omega^n(M) \xrightarrow{d_{Tf}} 0$$

Associated to this complex, for each $k = 0, \dots, n$, we have the k-th cohomology space

$$H^k_{Tf,\mathrm{dR}}(M;\mathbb{R}) := \frac{\ker d_{Tf}|_{\Omega^k(M)}}{\mathrm{Im}\; d_{Tf}|_{\Omega^{k-1}(M)}}$$

As it turns out, the information contained in the dimensionality of the cohomology spaces is the same in the de Rham complex and in the deformation:

Proposition 2.1. *For any* $0 \le k \le n$*,*

$$\dim H^k_{Tf,dR}(M;\mathbb{R}) = \dim H^k_{dR}(M;\mathbb{R}) \tag{2.2}$$

Proof. This is a mere consequence of the fact that $d_{Tf}e^{-Tf} = e^{-Tf}d$. We claim that the linear map $\Phi : \Omega^k(M) \to \Omega^k(M)$ given by

$$\alpha \mapsto e^{-Tf}\alpha$$

induces a linear map $H^k_{\mathrm{dR}}(M;\mathbb{R}) \to H^k_{Tf,\mathrm{dR}}(M;\mathbb{R})$. To see that, take a closed form $\alpha \in \Omega^k(M)$. Then

$$d_{Tf}e^{-Tf}\alpha = e^{-Tf}d\alpha = 0$$

That is, under Φ, $\ker d|_{\Omega^k(M)}$ is mapped into $\ker d_{Tf}|_{\Omega^k(M)}$. Also if $\beta \in \Omega^{k-1}(M)$ we have

$$\Phi(d\beta) = e^{-Tf}d\beta = d_{Tf}(e^{-Tf}\beta)$$

So Φ maps $\mathrm{Im}\; d|_{\Omega^{k-1}(M)}$ into $\mathrm{Im}\; d_{Tf}|_{\Omega^{k-1}(M)}$ and, therefore induces a linear map in the quotient $H^k_{\mathrm{dR}}(M;\mathbb{R}) \to H^k_{Tf,\mathrm{dR}}(M;\mathbb{R})$.

By doing a completely analogous resoning we can see that the map given by

$$\alpha \in \Omega^k(M) \mapsto e^{Tf}\alpha \in \Omega^k(M)$$

induces a linear map $H^k_{Tf,\mathrm{dR}}(M;\mathbb{R}) \to H^k_{\mathrm{dR}}(M;\mathbb{R})$ which is clearly the inverse of the map induced by Φ. So $H^k_{\mathrm{dR}}(M;\mathbb{R})$ and $H^k_{Tf,\mathrm{dR}}(M;\mathbb{R})$ are isomorphic and, in particular, have same dimension. $\square$

It is not hard to see that we can develop the Hodge Theory associated to the elliptic complex $(\Omega^\bullet(M), d_{Tf})$ in the same way as in the de Rham complex. Given $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$ we have

$$\langle d_{Tf}\alpha, \beta\rangle = \langle e^{-Tf} d\, e^{Tf}\alpha, \beta\rangle = \langle d\, e^{Tf}\alpha, e^{-Tf}\beta\rangle = \langle \alpha, e^{Tf} d^* e^{-Tf}\beta\rangle$$

In other words, the formal adjoint of d_{Tf} is

$$d^*_{Tf} = e^{Tf} d^* e^{-Tf} \tag{2.3}$$

Associated with these operators we can put $D_{Tf} = d_{Tf} + d^*_{Tf}$ so the corresponding Laplace operator for $(\Omega^\bullet(M), d_{Tf})$ is

$$D^2_{Tf} = d_{Tf} d^*_{Tf} + d^*_{Tf} d_{Tf}$$

From (2.1) and (2.3) it is clear that $D^2_{Tf} : \Omega^k(M) \to \Omega^k(M)$. The decomposition theorem then follows the same way as mentioned for the de Rham complex. In particular we have the analogous result to (1.2), which, according to (2.2) implies

$$\dim\ker D^2_{Tf}|_{\Omega^k(M)} = \dim H^k_{Tf,\mathrm{dR}}(M;\mathbb{R}) = \dim H^k_{\mathrm{dR}}(M;\mathbb{R}) \tag{2.4}$$

Thus we see that we reduced the problem of estimating the Betti numbers $\beta_k = \dim H^k_{\mathrm{dR}}(M;\mathbb{R})$ to analyzing the behavior of the kernel of D^2_{Tf}. By the independence of the parameter T in the right-hand side of (2.4), we might do that in the limit $T \to +\infty$. We will start by understanding $\ker D^2_{Tf}$ in a neighborhood of a critical point.

3 Local description of $\ker D^2_{Tf}$

Assume we have local coordinates $y = (y^1, \ldots, y^n)$ in a neighborhood U_x of a critical point of f, $x \in M$ like in Theorem 0.1. Assume the metric is such that on U_x the vectors $e_i = \frac{\partial}{\partial y^i}$ form an oriented orthonormal basis. From Theorem 0.1 we see that;

$$df = -y^1\, dy^1 - \cdots - y^{n_f(x)}\, dy^{n_f(x)} + y^{n_f(x)+1}\, dy^{n_f(x)+1} + \cdots + y^n\, dy^n$$

Notice also that from (1.3) we have

$$\begin{aligned} d_{Tf} &= e^{-Tf}\left(\sum_{i=1}^{n} dy^i \wedge \nabla_{e_i}\right) e^{Tf} = e^{-Tf}e^{Tf}\sum_{i=1}^{n} dy^i \wedge \nabla_{e_i} + e^{-Tf}\sum_{i=1}^{n} dy^i \wedge d(e^{Tf})(e_i) \\ &= \sum_{i=1}^{n} dy^i \wedge \nabla_{e_i} + T\sum_{i=1}^{n} df(e_i) dy^i \wedge = d + Tdf\wedge \end{aligned}$$

Similarly, from (1.4) we obtain

$$d_{Tf}^* = \sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i} + T\sum_{i=1}^{n} df(e_i) e_i \lrcorner = d^* + T(df^*)\lrcorner$$

where $(df)^* \in C^\infty(M; TM)$ is the dual of df under g. In local coordinates, $df^* = (-y^1, \dots, -y^{n_f(x)}, y^{n_f(x)+1}, \dots, y^n)$. By adding the previous results we conclude

$$D_{Tf} = D + T\hat{c}(df)$$

Now we can calculate

$$D_{Tf}^2 = D^2 + T\left[D\hat{c}(df) + \hat{c}(df)D\right] + T^2\hat{c}(df)^2$$

By the properties of Clifford actions, the last term is simply $T^2\hat{c}(df)^2 = T^2|df|^2 = T^2|y|^2$ in local coordinates. The first term can be seen as

$$D^2 = \sum_{i,j=1}^{n} c(e_i)c(e_j)\nabla_{e_i}\nabla_{e_j} = \sum_i c(e_i)^2\nabla_{e_i}^2 = -\sum_i \nabla_{e_i}^2$$

But in local coordinates $\nabla_{e_i} = \frac{\partial}{\partial y^i}$, so $D^2 = -\sum_{i=1}^n \left(\frac{\partial}{\partial y^i}\right)^2$. The remaining term can be simplified by using the fact that ∇ is a Clifford connection:

$$\begin{aligned} T\left[D\hat{c}(df) + \hat{c}(df)D\right] &= T\left[\sum_{i=1}^{n} c(e_i)\nabla_{e_i}\hat{c}(df) + \sum_{i=1}^{n} \hat{c}(df)c(e_i)\nabla_{e_i}\right] \\ &= T\left[\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla_{e_i}df) + \sum_{i=1}^{n} c(e_i)\hat{c}(df)\nabla_{e_i} + \sum_{i=1}^{n} \hat{c}(df)c(e_i)\nabla_{e_i}\right] \\ &= T\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla_{e_i}df) \end{aligned}$$

Notice that for $i \leq n_f(x)$, $\nabla_{e_i} df = -dy^i$ and for $i > n_f(x)$, $\nabla_{e_i} df = dy^i$, thus

$$T\left[D\hat{c}(df) + \hat{c}(df)D\right] = -T\sum_{i=1}^{n_f(x)} c(e_i)\hat{c}(e_i) + T\sum_{i=n_f(x)+1}^{n} c(e_i)\hat{c}(e_i)$$

We can rewrite this by adding and subtracting $nT = T\sum_{i=1}^{n} 1$ thus obtaining

$$T\left[D\hat{c}(df) + \hat{c}(df)D\right] = T\left[\sum_{i=1}^{n_f(x)} (1 - c(e_i)\hat{c}(e_i)) + \sum_{i=n_f(x)+1}^{n} (1 + c(e_i)\hat{c}(e_i))\right] - nT$$

But from definition,

$$c(e_i)\hat{c}(e_i) = (dy^i\wedge - e_i\lrcorner)(dy^i\wedge + e_i\lrcorner) = dy^i\wedge e_i\lrcorner - e_i\lrcorner dy^i\wedge = 1 - 2e_i\lrcorner dy^i\wedge = 2dy^i\wedge e_i\lrcorner - 1$$

Thus

$$\sum_{i=1}^{n_f(x)} (1 - c(e_i)\hat{c}(e_i)) + \sum_{i=n_f(x)+1}^{n} (1 + c(e_i)\hat{c}(e_i)) = \sum_{i=1}^{n_f(x)} 2e_i\lrcorner dy^i \wedge + \sum_{i=n_f(x)+1}^{n} 2dy^i \wedge e_i\lrcorner$$

Now we can put all the terms together to get

$$D_{Tf}^2 = -\sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)^2 - nT + T^2|y|^2 + 2T\left[\sum_{i=1}^{n_f(x)} e_i\lrcorner dy^i \wedge + \sum_{i=n_f(x)+1}^{n} dy^i \wedge e_i\lrcorner\right] \quad (3.1)$$

Here we already can see some localizing aspects of Witten deformation. As T increases, it is necessary that a form in the kernel of D_{Tf}^2 be concentrated around $y = 0$, otherwise the term $T^2|y|^2$ would make it impossible for it to vanish.

It is not hard to see that the kernel of the last term of the expression 3.1, given by the linear operator

$$\sum_{i=1}^{n_f(x)} 2e_i\lrcorner dy^i \wedge + \sum_{i=n_f(x)+1}^{n} 2dy^i \wedge e_i\lrcorner$$

is generated by $dy^1 \wedge \cdots \wedge dy^{n_f(x)}$. On the other hand the first part, given by the differential operator

$$-\sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)^2 - nT + T^2|y|^2$$

acts only on the components of the form. So if we multiply $dy^1 \wedge \cdots \wedge dy^{n_f(x)}$ by a function $g(y)$ that is a solution to

$$\left(-\sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)^2 - nT + T^2|y|^2\right) g(y) = 0$$

we will have an element of the kernel of D^2_{Tf}. But this operator is a harmonic oscilator, with a one dimensional kernel and well known solution given by

$$g(y) = \exp\left(\frac{-T|y|^2}{2}\right)$$

For more details, see [1]. These calculations can be summarized by the following proposition:

Proposition 3.1. *For any $T > 0$, the operator*

$$-\sum_{i=1}^{n}\left(\frac{\partial}{\partial y^i}\right)^2 - nT + T^2|y|^2 + 2T\left[\sum_{i=1}^{n_f(x)} e_i \lrcorner dy^i \wedge + \sum_{i=n_f(x)+1}^{n} dy^i \wedge e_i \lrcorner\right]$$

acting on $\Omega(\mathbb{R}^n)$ is nonnegative. Its kernel is 1-dimensional and it is generated by

$$\exp\left(\frac{-T|y|^2}{2}\right) dy^1 \wedge \cdots \wedge dy^{n_f(x)} \tag{3.2}$$

All the nonzero eigenvalues of this operator are greater than CT for a fixed $C > 0$.

4 Global Description of $\ker D^2_{Tf}$

Proposition 3.1 gives the behavior of the kernel of D^2_{Tf} in the neighborhood of a critical point. We can globalize it by first extending the generating form (3.2) to M. In order to do that, let $\gamma(|y|)$ be a smooth bump function around $y = 0$. That is, $\gamma(|y|) = 1$ when $|y| \leq r$ and $\gamma(|y|) = 0$ when $|y| \geq 2r$ for some radius r such that the ball of radius $2r$ is still contained in U_x. Then we define $\rho_{x,T} \in \Omega^{n_f(x)}(M)$ by

$$\rho_{x,T} = \frac{\gamma(|y|)}{\sqrt{\alpha_{x,T}}} \exp\left(\frac{-T|y|^2}{2}\right) dy^1 \wedge \cdots \wedge dy^{n_f(x)}$$

where $\alpha_{x,T}$ is a normalization factor to make $\langle \rho_{x,T}, \rho_{x,T} \rangle = 1$. That is,

$$\alpha_{x,T} = \int_{U_x} \gamma(|y|)^2 \exp\left(-T|y|^2\right) dy^1 \cdots dy^n$$

Let us denote by

$$H^0 := H^0(M, \Lambda M) = L^2(M, \Lambda M) \quad \text{and} \quad H^1 := H^1(M, \Lambda M)$$

the Sobolev spaces of forms associated to the 0-th and 1-st norm $\|\cdot\|_0$ and $\|\cdot\|_1$ induced by the inner product (1.1). Denote by $Z(f)$ the set of critical points of f, let $E_T \subset H^0$ be the subspace generated by $\rho_{x,T}$ for all $x \in Z(f)$ and $E_T^\perp \subset H^0$ its orthogonal complement. We will decompose D_{fT} as operators between these spaces by using the orthogonal projections $p_T : H^0 \to E_T$ and $p_T^\perp : H^0 \to E_T^\perp$. Thus, denote

$$D_{T,1} = p_T D_{Tf} p_T \qquad D_{T,2} = p_T D_{Tf} p_T^\perp$$
$$D_{T,3} = p_T^\perp D_{Tf} p_T \qquad D_{T,4} = p_T^\perp D_{Tf} p_T^\perp$$

Estimates for each of these components can be separately done according to the following proposition.

Proposition 4.1. *(i) For any $T > 0$, $D_{T,1} = 0$*

(ii) There is a $T_1 > 0$ such that for any $s \in E_T^\perp \cap H^1$, $s' \in E_T$ and $T \geq T_1$ we have

$$\|D_{T,2}s\|_0 \leq \frac{\|s\|_0}{T} \qquad \text{and} \qquad \|D_{T,3}s'\|_0 \leq \frac{\|s'\|_0}{T}$$

(iii) There are $T_2 > 0$ and $C > 0$ such that for any $s \in E_T^\perp \cap H^1$ and $T \geq T_2$,

$$\|D_{Tf}s\|_0 \geq C\sqrt{T}\|s\|_0$$

Proof. (i) From the definition of E_T we know that

$$p_T s = \sum_{x \in Z(f)} \langle \rho_{x,T}, s\rangle_0 \rho_{x,T}$$

But since $\langle \rho_{x,T}, s\rangle_0 \rho_{x,T} \in \Omega^{n_f(x)}(M)$ and since $\rho_{x,T}$ and its derivatives have compact inside U_x, that means $D_{Tf}\left(\langle \rho_{x,T}, s\rangle_0 \rho_{x,T}\right) \in \Omega^{n_f(x)-1}(M) \oplus \Omega^{n_f(x)+1}(M)$ has compact inside U_x. But inside U_x, the projection p_T maps into $\Omega^{n_f(x)}(M)$, so

$$p_T D_{Tf}\left(\langle \rho_{x,T}, s\rangle_0 \rho_{x,T}\right) = 0$$

for each $x \in Z(f)$ and consequently $D_{T,1} = 0$.

(ii) Since D_{Tf} is self adjoint, we see that $D_{T,3}$ is the formal adjoint of $D_{T,2}$, so the second estimate follow from the first one. From the definitions, given $s \in E_T^\perp \cap H^1$

we have

$$D_{T,2}s = p_T D_{Tf} s = \sum_{x\in Z(f)} \langle \rho_{x,T}, D_{Tf}s\rangle_0 \rho_{x,T} = \sum_{x\in Z(f)} \langle D_{Tf}\rho_{x,T}, s\rangle_0 \rho_{x,T}$$

Thus

$$\|D_{T,2}s\|_0 \le \sum_{x\in Z(f)} \|D_{Tf}\rho_{x,T}\|_0 \|s\|_0$$

But

$$\|D_{Tf}\rho_{x,T}\|_0^2 = \int_{U_x} \left| D_{Tf} \frac{\gamma(|y|)}{\sqrt{\alpha_{x,T}}} \exp\left(\frac{-T|y|^2}{2}\right)\right|^2 dy^1 \cdots dy^n$$

The factor multiplying $\gamma(|y|)$ is on the kernel of D_{Tf}^2 so we only have to consider the action of D_{Tf} on $\gamma(|y|)$. But the derivatives of $\gamma(|y|)$ vanish everywhere except an annulus around x. That means if we can bound the terms depending on T by a term of the form e^{-RT}, with R the inner radius of the annulus. The remaining terms will be bounded by a constant. Since that same behavior happens for every $x \in Z(f)$ we conclude that there are constants C_1 and C_2 such that

$$\|D_{T,2}s\|_0 \le C_1 e^{-C_2 T} \|s\|_0$$

In particular, there is a $T_1 > 0$ such that $T \ge T_1$ gives us the required estimate.

(iii) The proof of this estimate involves the same kind of computations, but it is longer, requiring some particular cases to be analyzed separately. For sake of not making this text loose track of its main purpose by dwelling on technical details we will just refer the reader to [1]. □

For any $c > 0$, let $E_T(c) \subset H^0$ be the direct sum of eigenspaces of D_{Tf} corresponding to the eigenvalues in the interval $[-c, c]$ and let $P_T(c) : H^0 \to E_T(c)$ be the orthogonal projection onto $E_T(c)$. Using a little of spectral theory and the previous proposition one can obtain the following lemma (see [1]).

Lemma 4.2. *There are constants $C_1 > 0$ and $T_3 > 0$ such that for any $T \ge T_3$ and any $\sigma \in E_T$ we have*

$$\|P_T(c)\sigma - \sigma\|_0 \le \frac{C_1}{T}\|\sigma\|_0$$

Now we present the main result, from where the Morse inequalities will follow as a corollary.

Proposition 4.3. *For any $c > 0$, there is a $T_0 > 0$ such that for every $T \ge T_0$ the number of eigenvalues of $D_{Tf}^2|_{\Omega^i(M)}$, in $[0, c]$ equals to m_i, for $0 \le i \le n$.*

Proof. If we take $x_1 \neq x_2 \in Z(f)$ since $\langle \rho_{x_1,T}, \rho_{x_2,T} \rangle_0 = 0$ we see that by Cauchy-Shwarz inequality

$$\begin{aligned} |\langle P_T(c)\rho_{x_1,T}, P_T(c)\rho_{x_2,T}\rangle_0| \leq \|P_T(c)\rho_{x_1,T} - \rho_{x_1,T}\|_0 \|P_T(c)\rho_{x_2,T} - \rho_{x_2,T}\|_0 \\ + \|P_T(c)\rho_{x_1,T} - \rho_{x_1,T}\|_0 + \|P_T(c)\rho_{x_2,T} - \rho_{x_2,T}\|_0 \end{aligned}$$

So, by applying Lemma 4.2 to the $\rho_{x_i,T}$ we have $\langle P_T(c)\rho_{x_1,T}, P_T(c)\rho_{x_2,T}\rangle_0 \to 0$ when $T \to \infty$. In particular, we conclude that for large enough T all the $P_T(c)\rho_{x,T}$ are linearly independent. That means there must be a $T_5 > 0$ such that $T > T_5$ implies

$$\dim E_T(c) \geq \dim P_T(c)E_T = \dim E_T \tag{4.1}$$

Assume we have $\dim E_T(c) > \dim E_T$. That means there is a nonzero element $s \in E_T(c)$ perpendicular to $P_T(c)E_T$. Thus, for every $x \in Z(f)$

$$\langle s, P_T(c)\rho_{x,T}\rangle_0 = 0$$

This in turn implies

$$\begin{aligned} p_T s &= \sum_{x\in Z(f)} \langle s, \rho_{x,T}\rangle_0 \rho_{x,T} - \sum_{x\in Z(f)} \langle s, P_T(c)\rho_{x,T}\rangle_0 P_T(c)\rho_{x,T} \\ &= \sum_{x\in Z(f)} \langle s, \rho_{x,T}\rangle_0 (\rho_{x,T} - P_T(c)\rho_{x,T}) - \sum_{x\in Z(f)} \langle s, \rho_{x,T} - P_T(c)\rho_{x,T}\rangle_0 P_T(c)\rho_{x,T} \end{aligned}$$

Once again, by applying Cauchy-Schwarz in the inner products and using Lemma 4.2 we see that there is a constant $C_2 > 0$ such that $T > T_5$ implies

$$\|p_T s\|_0 \leq \frac{C_2}{T}\|s\|_0$$

So that means we have

$$\|p_T^\perp s\|_0 = \|s - p_T s\|_0 \geq \|s\|_0 - \|p_T s\|_0 \geq C_3\|s\|_0$$

for some constant C_3. Thus, Proposition 4.1 implies that

$$\begin{aligned} C\sqrt{T}C_3\|s\|_0 \leq C\sqrt{T}\|p_T^\perp s\|_0 \leq \|D_{Tf}p_T^\perp s\|_0 = \|D_{Tf}s - D_{Tf}p_T s\|_0 \\ \leq \|D_{Tf}s\|_0 + \|D_{T,3}s\|_0 \leq \|D_{Tf}s\|_0 + \frac{\|s\|_0}{T} \end{aligned}$$

Notice that above we used the fact that $D_{Tf}p_Ts = D_{T1}s + D_{T3}s = D_{T3}s$, also by Proposition 4.1. Rewriting the previous estimate we get

$$\|D_{Tf}s\|_0 \geq CC_3\sqrt{T}\|s\|_0 - \frac{1}{T}\|s\|_0$$

So if $s \neq 0$ as $T \to \infty$ we see that $\|D_{Tf}s\|_0 \to \infty$, which contradicts the assumption that $s \in E_T(c)$, that is, s is a linear combination of eigenvectors of D_{Tf} with eigenvalues bounded in $[-c, c]$.

Thus we can conclude

$$\dim E_T(c) = \dim E_T = \sum_{i=0}^{n} m_i$$

and $\{P_T(c)\rho_{x,T}\}_{x\in Z(f)}$ form a basis for $E_T(c)$.

Now we will analyze each order $0 \leq i \leq n$ separately by decomposing $E_T(c)$. Let Q_i be the orthogonal projection from H^0 to onto the completion of $\Omega^i(M) \subset H^0$. By Lemma 4.2 we see that

$$\|Q_{n_f(x)}P_T(c)\rho_{x,T} - \rho_{x,T}\|_0 \leq \|P_T(c)\rho_{x,T} - \rho_{x,T}\|_0 \leq \frac{C_1}{T}\|\rho_{x,T}\|_0 = \frac{C_1}{T}$$

for any $x \in Z(f)$.So, by the same argument leading to (4.1), we conclude that for large enough T, the $Q_{n_f(x)}P_T(c)\rho_{x,T}$ are linearly independent, so there must be $T_0 > 0$ such that $T \geq T_0$ implies

$$\dim Q_iE_T(c) \geq m_i$$

If it were the case that for some i, $\dim Q_iE_T(c) > m_i$, then we would have

$$\sum_{i=0}^{n} \dim Q_iE_T(c) > \sum_{i=0}^{n} m_i = \dim E_T(c)$$

which is a contradiction. Therefore, for any $0 \leq i \leq n$ and $T \geq T_0$ we must have

$$\dim Q_iE_T(c) = m_i \tag{4.2}$$

Now recall that D^2_{Tf} preserves the grading of $\Omega(M)$, so if s is an eigenvector of D_{Tf} with eigenvalue $\mu \in [-c, c]$ then

$$D^2_{Tf}Q_is = Q_iD^2_{Tf}s = \mu^2 Q_is$$

That is, $Q_iE_T(c)$ is the space of eigenvectors of $D^2_{Tf}|_{\Omega^i(M)}$ with eigenvalues in $[0, c]$.

The proposition then follows from (4.2). □

5 Proof of Morse Inequalities

Finally we can prove the Morse Inequalities (Theorem 0.2) as a consequence of Proposition 4.3.

For any $0 \leq i \leq n$ denote by $F_{Tf,i}^{[0,c]} \subset \Omega(M)$ the vector space generated by the eigenspaces of $D_{Tf}^2|_{\Omega^i(M)}$ with eigenvalues in $[0,c]$. We will assume T large enough so that by Proposition 4.3 $\dim F_{Tf,i}^{[0,c]} = m_i$. From

$$d_{Tf} D_{Tf}^2 = d_{Tf} d_{Tf}^* d_{Tf} = D_{Tf}^2 d_{Tf}$$

and

$$d_{Tf}^* D_{Tf}^2 = d_{Tf}^* d_{Tf} d_{Tf}^* = D_{Tf}^2 d_{Tf}^*$$

we see that $d_{Tf} : F_{Tf,i}^{[0,c]} \to F_{Tf,i+1}^{[0,c]}$ and $d_{Tf} : F_{Tf,i}^{[0,c]} \to F_{Tf,i-1}^{[0,c]}$, so we have a finite dimensional subcomplex of $(\Omega^\bullet(M), d_{Tf})$ given by

$$(F_{Tf,\bullet}^{[0,c]}, d_{Tf}) : 0 \to F_{Tf,0}^{[0,c]}(M) \xrightarrow{d_{Tf}} F_{Tf,1}^{[0,c]} \xrightarrow{d_{Tf}} \cdots \xrightarrow{d_{Tf}} F_{Tf,n}^{[0,c]} \xrightarrow{d_{Tf}} 0$$

By applying the Hodge decomposition theorem to this subcomplex we have that the dimension of the i-th comohomology space, given by

$$\frac{\ker d_{TF}|_{F_{Tf,i}^{[0,c]}}}{\mathrm{Im}\ d_{TF}|_{F_{Tf,i-1}^{[0,c]}}}$$

is equal to the dimension of the kernel of the associated Laplace operator $D_{Tf}^2|_{F_{Tf,i}^{[0,c]}}$. But since $\ker D_{Tf}^2|_{\Omega^i(M)} \subset F_{Tf,i}^{[0,c]}$ we see that $\ker D_{Tf}^2|_{F_{Tf,i}^{[0,c]}} = \ker D_{Tf}^2|_{\Omega^i(M)}$. Therefore

$$\beta_i = \dim\left(\ker D_{Tf}^2|_{\Omega^i(M)}\right) = \dim\left(\ker D_{Tf}^2|_{F_{Tf,i}^{[0,c]}}\right) \leq \dim F_{Tf,i}^{[0,c]} = m_i$$

which are the weak Morse inequalities.

Also, from basic finite-dimensional linear algebra we have

$$
\begin{aligned}
m_i = \dim F_{Tf,i}^{[0,c]} &= \dim\left(\ker d_{Tf}|_{F_{Tf,i}^{[0,c]}}\right) + \dim\left(\text{Im } d_{Tf}|_{F_{Tf,i}^{[0,c]}}\right) \\
&= \dim\left(\frac{\ker d_{TF}|_{F_{Tf,i}^{[0,c]}}}{\text{Im } d_{TF}|_{F_{Tf,i-1}^{[0,c]}}}\right) + \dim\left(\text{Im } d_{Tf}|_{F_{Tf,i-1}^{[0,c]}}\right) + \dim\left(\text{Im } d_{Tf}|_{F_{Tf,i}^{[0,c]}}\right) \\
&= \beta_i + \dim\left(\text{Im } d_{Tf}|_{F_{Tf,i-1}^{[0,c]}}\right) + \dim\left(\text{Im } d_{Tf}|_{F_{Tf,i}^{[0,c]}}\right)
\end{aligned}
$$

Thus, by adding all m_j with alternated signs we see that

$$\sum_{j=0}^{i}(-1)^j m_{i-j} = \sum_{j=0}^{i}(-1)^j \beta_{i-j} + \dim\left(\text{Im } d_{Tf}|_{F_{Tf,i}^{[0,c]}}\right)$$

In particular, for $0 \leq i \leq n$ we get

$$\sum_{j=0}^{i}(-1)^j \beta_{i-j} \leq \sum_{j=0}^{i}(-1)^j m_{i-j}$$

and for $i = n$, since $\text{Im } d_{Tf}|_{F_{Tf,n}^{[0,c]}} = 0$:

$$\sum_{j=0}^{n}(-1)^j m_{n-j} = \sum_{j=0}^{n}(-1)^j \beta_{n-j}$$

Which are the strong Morse inequalities.

References

[1] Weiping Zhang. "Lectures on Chern-Weil Theory and Witten Deformations," World Scientific, (2001).

[2] J. Milnor. "Morse Theory," Princeton University Press (1963).

[3] E. Witten. "Supersymmetry and Morse theory," J. Diff. Geom. (1982), 661-692.

[4] C. Kottke. "Linear Analysis on Manifolds", Notes for Math 7376, Spring 2016. http://www.northeastern.edu/ckottke/7376_sp16/lacm.pdf.