WHAT IS L^2-COHOMOLOGY AND WHY SHOULD YOU CARE?

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1. What is L^2-cohomology?

L^2-cohomology may be thought of as a slight modification of the ordinary de Rham cohomology of a smooth manifold. Let (M, g) be a Riemannian manifold of dimension n, and let $(C^\infty(M; \Lambda^\bullet), d^\bullet)$ be the de Rham complex of M; i.e., $C^\infty(M; \Lambda^k)$ consists of the space of smooth k-forms on M, and

$$d^k = d : C^\infty(M; \Lambda^k) \to C^\infty(M; \Lambda^{k+1})$$

is the exterior derivative of differential forms. Then, the well-known de Rham Theorem says that the cohomology of this complex (called, obviously, the de Rham cohomology of M) is isomorphic to the singular cohomology of M, with real coefficients:

$$H^k_{dR}(M) \cong H^k(M; \mathbb{R}).$$

The main point is then that the de Rham cohomology groups are *topological* data about the manifold.

Since M has a Riemannian metric, we can add some geometry to this picture. The metric g induces an L^2-metric on the spaces $C^\infty(M; \Lambda^k)$, given by

$$(\alpha, \beta)_{L^2} := \int_M \langle \alpha, \beta \rangle \, \omega_g,$$

where $\langle \alpha, \beta \rangle(x) = \langle \alpha_x, \beta_x \rangle$ is the usual inner product on forms, and $\omega_g \in C^\infty(M; \Lambda^n)$ is the volume form. We may then consider the L^2-completion, $L^2(M; \Lambda^k)$, of $C^\infty(M; \Lambda^k)$ with respect to the L^2-metric. We define d to be the exterior differential with domain

$$\operatorname{dom} d = \{\alpha \in \Omega^k_{(2)}(M; \mathbb{R}) | d\alpha \in \Omega^{k+1}_{(2)}(M; \mathbb{R})\}.$$

Where

$$\Omega^k_{(2)}(M; \mathbb{R}) := C^\infty(M; \Lambda^k) \cap L^2(M; \Lambda^k).$$

Definition 1.1. *The L^2-cohomology groups of M are the cohomology groups of the complex $(\Omega^\bullet_{(2)}(M; \mathbb{R}), d^\bullet)$:*

$$H^i_{(2)}(M; \mathbb{R}) := \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}.$$

The L^2-cohomology groups of M are no longer topological invariants–they depend only on the *quasi-isometry* class of the metric g.

Definition 1.2. *Two Riemannian metrics g and h on M are called quasi-isometric if there is a positive constant K such that, for all $x \in M$,*

$$K^{-1} g_x \leq h_x \leq K g_x.$$

It then follows that, if g and h are quasi-isometric Riemannian metrics on M, an i-form α on M is square-integrable with respect to the metric g if and only if it is square-integrable with respect to h.

Let $\star : C^\infty(M; \Lambda^k) \to C^\infty(M; \Lambda^{n-k})$ be the Hodge star operator, which is the unique operator satisfying

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega_g$$

for all $\alpha \in C^\infty(M; \Lambda^k)$. The star operator allows us to define the codifferential map, $\delta_k : C^\infty(M; \Lambda^k) \to C^\infty(M; \Lambda^{k-1})$, given by $\delta_k = (-1)^{n(k+1)+1} \star d \star$.

Let d_0^k be the restriction of d^k to the space of compactly supported smooth k-forms, and similarly define $\delta_{k,0}$. We again slightly change the domain of δ_k, to be

$$\operatorname{dom} \delta_k = \{\alpha \in \Omega^k_{(2)}(M; \mathbb{R}) | \delta_k \alpha \in \Omega^{k-1}_{(2)}(M; \mathbb{R})\}$$

Since $\operatorname{dom} \delta_{k+1,0} = C_c^\infty(M; \Lambda^{k+1})$ is dense, and by Stokes' Theorem,

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$$

whenever $\alpha \in \operatorname{dom} d^k$ and $\beta \in \operatorname{dom} \delta_{k+1,0}$. It follows then that d^k has a well-defined weak closure $\delta^*_{k+1,0}$, as well as a strong closure $\overline{d^k}$. This means that $\alpha \in \operatorname{dom} \overline{d^k}$ and $\overline{d^k}\alpha = \eta$ if there is a sequence $\alpha_j \in \operatorname{dom} d^k$ such that $\alpha_j \to \alpha$ and $d\alpha_j \to \eta$ in L^2. Similarly, δ_k has strong closure $\overline{\delta_k}$.

Then, we can also define

$$H^k_{(2),\sharp}(M; \mathbb{R}) = \operatorname{Ker} \overline{d^k} / \operatorname{Im} \overline{d^{k-1}}$$

as a possible candidate for L^2-cohomology. It is shown in [C] that by Elliptic regularity, the natural morphism

$$\iota_{(2)} : H^k_{(2)}(M; \mathbb{R}) \to H^k_{(2),\sharp}(M; \mathbb{R})$$

is *always* an isomorphism, so one can use either definition. We also have natural pseudonorms on the spaces $H^k_{(2)}(M; \mathbb{R}), H^k_{(2),\sharp}(M; \mathbb{R})$, given by

$$\|U\| = \inf_{\alpha \in U} \|\alpha\|$$

which are preserved by the above isomorphism. Since $\overline{d^k}$ is a closed operator, the Open Mapping Theorem implies that the image of d^k is closed if $H^k_{(2)}(M; \mathbb{R}) = H^k_{(2),\sharp}(M; \mathbb{R})$ is finite dimensional.

Example 1.3. Let $M = \mathbb{R}$, with its standard metric. It is easy to see that $H^0_{(2)}(\mathbb{R})$ is 0, since a constant function is L^2 if and only if it is the zero function. $H^1_{(2)}(\mathbb{R})$ is a bit more complicated.

If f is a smooth function such that $f(x) = 1/x$ for $|x| \geq 1$, then we trivially have $f\,dx \in \operatorname{Ker} \overline{d^1}$. If $f\,dx = d\alpha$ for some smooth function α, we we must have $\alpha(x) = \log(x) + C$ for $x > 1$. But then, $\alpha \notin L^2(\mathbb{R})$, which implies $f\,dx \notin \operatorname{Im} d^0$. As $H^1_{(2)}(\mathbb{R}) \cong H^1_{(2),\sharp}(\mathbb{R})$, we also know that $f\,dx \notin \operatorname{Im} \overline{d^0}$ (else $f\,dx$ would be zero in $H^1_{(2),\sharp}(\mathbb{R})$). Let ϕ be a smooth function supported on $[-2, 2]$, such that $\phi_{|_{[-1,1]}} = 1$, and set $\phi_n(x) = \phi(x/n)$. Then, we have $d(\phi_n\alpha) \to f\,dx$ in L^2, so that $\operatorname{Im} d^0$ is not closed. By the above remark, this implies that $H^1_{(2)}(\mathbb{R})$ is infinite dimensional!

2. Why should I care?

What is the point of any of this? One answer is to attempt to extend success of Hodge theory for compact oriented Riemannian manifolds to less well-behaved spaces. We first review some basic Hodge theory, then examine the results of Cheeger [C] and Cheeger, Goresky, and MacPherson in [CGM].

Definition 2.1. *The **Laplacian** is the second order differential operator*

$$\Delta = d\delta + \delta d$$

*on the spaces $C^\infty(M; \Lambda^k)$, where $\delta = (-1)^{n(k-1)+1} \star d \star$ is the formal adjoint of the exterior derivative d. A k-form α is called **Harmonic** if $\Delta\alpha = 0$.*

Recall that, on a compact Riemannian manifold, the harmonic forms are those $\alpha \in C^\infty(M; \Lambda^k)$ such that α is both closed (i.e., $d\alpha = 0$) and coclosed (i.e., $\delta\alpha = 0$). The famous Hodge Theorem then tells that every de Rham cohomology class has a unique harmonic representative; i.e., the de Rham cohomology is isomorphic to the space of harmonic forms. Moreover, if α is a harmonic form, then $\star\alpha$ is also a harmonic form. Indeed, a quick calculation shows that

$$\begin{aligned}
\Delta(\star\alpha) &= [d \star d \star (\star\alpha) + \star d \star d(\star\alpha)] \\
&= \Big[(-1)^{k(n+k)+1} d \star (d\alpha) + \star d(\delta\alpha)\Big] = 0
\end{aligned}$$

since α is both closed and coclosed. This observation then yields an isomorphism

$$\begin{aligned}
H^k(M; \mathbb{R}) &\xrightarrow{\sim} H^{n-r}(M; \mathbb{R}) \\
\alpha &\mapsto \star\alpha
\end{aligned}$$

which is seen to be none other than Poincaré duality.

In the above case, where M is compact, L^2-cohomology is naturally isomorphic to de Rham cohomology, so all these constructions hold for the groups $H^k_{(2)}(M)$. What if M is not compact? Clearly, we still have the Hodge star operator and the Laplacian. Only now, it is not the case that a harmonic form $\alpha \in \Omega^k_{(2)}(M;\mathbb{R})$ is necessarily closed and coclosed.

Example 2.2. Consider the function $f = x^2 - y^2$ defined on the open unit disk in $\mathbb{R}^2$. Then, we clearly have $\Delta f = 0$, so f is a harmonic 0-form and is square integrable, but f is not closed, as it is not a constant function. f is, however, coclosed, since $d \star f = 0$.

Consequently, harmonic forms may not define cohomology classes in L^2-cohomology. However, if we consider the subspace $\mathcal{H}^k(M)$ of closed and coclosed harmonic forms in $\Omega^k_{(2)}(M;\mathbb{R})$, there is a natural morphism, called the Hodge map

$$\iota_{\mathcal{H}} : \mathcal{H}^k(M) \to H^k_{(2)}(M;\mathbb{R})$$

which is not necessarily an isomorphism. When $\iota_{\mathcal{H}}$ *is* an isomorphism, we say that the *Strong Hodge Theorem* holds for M (again, this property only depends on the quasi-isometry class of the metric on M). The *surjectivity* of $\iota_{\mathcal{H}}$ is equivalent to $\overline{\operatorname{Im} d_0^{k-1}} \subseteq \operatorname{Im} \overline{d^{k-1}}$, and if Stokes' Theorem holds in the L^2 sense, then $\iota_{\mathcal{H}}$ is injective. This just means that

$$\langle \overline{d}\alpha, \beta\rangle = \langle \alpha, \overline{\delta}\beta\rangle$$

for all $\alpha \in \operatorname{dom} \overline{d}$ and $\beta \in \operatorname{dom} \overline{\delta}$, or equivalently, for all $\alpha \in \operatorname{dom} d$ and $\beta \in \operatorname{dom} \delta$. This property holds when M is complete (a result of M. Gaffney), but this is not the case we will be most interested in.

We then have the following result of Kodaira:

Theorem 2.3 (Kodaira)**.**

$$L^2(M;\Lambda^k) = \overline{\operatorname{Im} \delta_{k+1,0}} \oplus \overline{\operatorname{Im} d_0^{k-1}} \oplus \mathcal{H}^k(M)$$

where the sum is orthogonal and preserves $\Omega^k_{(2)}(M;\mathbb{R})$*. This result is sometimes known as the* Weak Hodge Theorem*, and holds for M an arbitrary incomplete Riemannian manifold.*

From this theorem, if $\alpha \in \operatorname{Ker} \overline{d^k}$, we have $\alpha \in \overline{\operatorname{Im} d_0^{k-1}} \oplus \mathcal{H}^k(M)$. Indeed, Theorem 2.3 implies that, if $\overline{d^k}\alpha = 0$ and $\alpha \in \overline{\operatorname{Im} \delta_{k+1,0}}$, there exist sequences $\alpha_{1,j} \in \operatorname{dom} d^k$ and $\alpha_{2,j} \in \operatorname{Im} \delta_{k+1,0}$ such that $\alpha_{1,j} \to \alpha$, $d\alpha_{1,j} \to 0$, and $\delta\alpha_{2,j} \to \alpha$. Then,

$$(\alpha_{1,j}, \delta\alpha_{2,j})_{L^2} = (d\alpha_{1,j}, \alpha_{2,j})_{L^2} \to \|\alpha\|^2_{L^2} = 0$$

so that $\alpha = 0$. From this, it follows that the Strong Hodge Theorem is equivalent to $\overline{\operatorname{Im} d_0^{k-1}} = \operatorname{Im} \overline{d^{k-1}}$.

Perhaps the whole point of this discussion is to understand the difficulties that may arise from extending Hodge theory on a compact Riemannian manifold to an incomplete Riemannian manfiold. Secondary to this, in the absence of the Strong Hodge Theorem holding, extracting any topological interpretation out of the spaces $\mathcal{H}^k(M)$ and $H^k_{(2)}(M;\mathbb{R})$.

3. L^2-Cohomology of Metric Cones

We now want to describe the L^2-cohomology for the simplest singularity in the compact case. We spend a good deal of time on this case, as it paves the way for the calculation of L^2-cohomology in many other, more general cases. Let M be an n-dimensional (oriented) compact Riemannian manifold, with metric g_M. Then, the *metric cone* $C^*(M)$ is by definition the completion of the smooth incomplete Riemannian manifold $C(N) = (0,1)\times M$, with metric

$$g = dt\otimes dt + t^2\pi^* g_M$$

where t is the standard coordinate on $\mathbb{R}^+$, and $\pi : \mathbb{R}^+\times M \to M$ is the natural projection. Any differential k-form ξ on $C^*(M)$ can be written uniquely as

$$\xi = \eta + dt\wedge\zeta$$

where η and ζ are forms which do not involve dt. That is, with respect to local coordinates $(x) = (x_1,\cdots,x_n)$ on M, we can write

$$\eta(t,x) = \sum_{\alpha\in I(k)} \eta_\alpha(t,x)\,dx^\alpha$$

where $I(k)$ is the set of all multi-indices $\alpha = (\alpha_1,\cdots,\alpha_k)$ such that $1\le\alpha_1 < \cdots < \alpha_k \le n$, $dx^\alpha = dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_n}$, and $\eta_\alpha(t,x)$ are smooth functions on $\mathbb{R}^+\times M$. Similarly,

$$\eta(t,x) = \sum_{\alpha\in I(k-1)} \eta_\alpha(t,x)\,dx^\alpha.$$

Consequently, for fixed values of t, $\eta(t,x)$ and $\zeta(t,x)$ define differential forms on M. The metric on $C^*(M)$ is defined in such a way that

$$\|\xi\|^2 = t^{-2k}\|\eta(t,x)\|^2_M + t^{-2(k-1)}\|\zeta(t,x)\|^2_M.$$

We see this easily by checking on decomposable forms, and the fact that $g^{-1} = \partial_t\otimes\partial_t + t^{-2}\pi^* g_M^{-1}$. Then, we have the following theorem.

Theorem 3.1 (Cheeger). *Let M be a compact Riemannian manifold of dimension n, and let $C^*(M)$ be the metric cone on M. Then,*

$$H^k_{(2)}(C^*(M);\mathbb{R}) \cong \begin{cases} H^k(M;\mathbb{R}), & \text{if } k\le n/2\\ 0 & \text{if } k > n/2\end{cases}$$

We give a sketch of the proof of Theorem 3.1 (see [K],[C]):

If $\omega\in C^\infty(M;\Lambda^k)$, then with respect to local coordinates $(x_1,\cdots,x_n)$ on M, we can write

$$\omega(x) = \sum_{\alpha\in I(k)} \omega_\alpha(x)\,dx^\alpha.$$

The k-form $\pi^*\omega$ on $C^*(M)$ is then given in local coordinates $(t, x_1, \cdots, x_n)$ by the same formula:

$$\pi^*\omega(t,x) = \sum_{\alpha \in I(k)} \omega_\alpha(x)\, dx^\alpha,$$

so that

$$\|\pi^*\omega(t,x)\|^2 = t^{-2k}\|\omega(x)\|_M^2.$$

The volume form on $C^*(M)$ differs from M by a factor of t^n, so that

$$\int_{C^*(M)} \|\pi^*\omega\|^2 = \int_0^1 \int_M t^{-2k}\|\omega\|_M^2 t^n\, dt$$

Since M is compact, $\pi^*\omega$ is square-integrable on $C^*(M)$ if and only if $\omega = 0$ or

$$\int_0^1 t^{n-2k}\, dt < \infty.$$

Hence, if $k \leq n/2$, π^* restricts to a map

$$\pi^* : L^2(M; \mathbb{R}) \to L^2(C^*(M); \mathbb{R})$$

which commutes with d; thus, π^* induces a natural map

$$\pi^* : H^k(M; \mathbb{R}) \cong H^k_{(2)}(M; \mathbb{R}) \to H^k_{(2)}(C^*(M); \mathbb{R}).$$

We wish to show that this map is an isomorphism for all $k \leq m/2$, which we will do by constructing a chain homotopy. Given a k-form ξ on $C^*(M)$, we write $\xi = \eta + dt \wedge \zeta$, where η and ζ don't involve dt, as before. We can then define the k-form $\frac{\partial \eta}{\partial t}(t,x)$ and $(k-1)$-form $\frac{\partial \zeta}{\partial t}(t,x)$ on $C^*(M)$ via

$$\frac{\partial \eta}{\partial t}(t,x) = \sum_{\alpha \in I(k)} \frac{\partial \eta_\alpha}{\partial t}(t,x)\, dx^\alpha$$

$$\frac{\partial \zeta}{\partial t}(t,x) = \sum_{\alpha \in I(k-1)} \frac{\partial \zeta_\alpha}{\partial t}(t,x)\, dx^\alpha.$$

We define the "differential" $d_M : C^\infty(C^*(M); \Lambda^k) \to C^\infty(C^*(M); \Lambda^{k+1})$ in the local coordinates $(x) = (x_1, \cdots, x_n)$ by

$$\begin{aligned} d_M\xi(t,x) = &\sum_{1 \leq j \leq n} \sum_{\alpha \in I(k)} \frac{\partial \eta_\alpha}{\partial t}(t,x)\, dx_j \wedge dx^\alpha \\ &+ \sum_{1 \leq j \leq n} \sum_{\alpha \in I(k-1)} \frac{\partial \zeta_\alpha}{\partial t}(t,x)\, dx_j \wedge dt \wedge dx^\alpha. \end{aligned}$$

Then,

$$d_M\xi = d_M\eta - dt \wedge d_M\zeta,$$

and

$$d\xi = d_M\xi + dt \wedge \frac{\partial \eta}{\partial t} = d_M\eta + dt \wedge \left(\frac{\partial \eta}{\partial t} - d_M\zeta\right).$$

For fixed $s \in (0,1)$, define a map

$$H : C^\infty(C^*(M);\Lambda^k) \to C^\infty(C^*(M);\Lambda^{k-1})$$

in the local coordinates $(x_1, \cdots, x_n)$ by

$$(H\xi)(t,x) = \sum_{\alpha \in I(k-1)} \left(\int_s^t \zeta_\alpha(\tau, x)\, d\tau \right) dx^\alpha$$

or, more compactly, as $H\xi = \int_s^t \zeta$. Then,

$$\begin{aligned} dH\xi &= d_M \int_s^t \zeta + dt \wedge \frac{\partial}{\partial t} \int_s^t \zeta \\ &= \int_s^t d_M\zeta + dt \wedge \zeta, \end{aligned}$$

and

$$\begin{aligned} Hd\xi &= H\left(d_M\eta + dt \wedge \left(\frac{\partial \eta}{\partial t} - d_M\zeta \right) \right) \\ &= \int_s^t \left(\frac{\partial \eta}{\partial t} - d_M\zeta \right) \\ &= \eta - \pi^*\eta^{(s)} - \int_s^t d_M\zeta, \end{aligned}$$

where $\eta^{(s)}(x) = \eta(s,x) \in C^\infty(M;\Lambda^k)$ for fixed $s \in (0,1)$. Consequently,

$$\begin{aligned} dH\xi + Hd\xi &= dt \wedge \zeta + \eta - \pi^*\eta^{(s)} \\ &= \xi - \pi^*\eta^{(s)}. \end{aligned}$$

As we are, we have constructed a chain homotopy between the complexes $C^\infty(M;\Lambda^\bullet)$ and $C^\infty(C^*(M);\Lambda^\bullet)$, at least for $\bullet \le n/2$.

If $\xi \in \Omega^k_{(2)}(M;\mathbb{R})$, then the integral

$$\int_{C^*(M)} \|\xi\|^2 = \int_0^1 \int_M \left(t^{-2k}\|\eta^{(t)}\|_M + t^{-2(k-1)}\|\zeta^{(t)}\|_M \right) t^n\, dt$$

is finite. $H\xi$ is a $(k-1)$-form, so

$$\int_{C^*(M)} \|H\xi\|^2 \le \int_0^1 \int_M t^{-2(k-1)} \int_s^t \|\zeta^{(\tau)}\, d\tau\|_M^2 t^n\, dt.$$

Via the Cauchy-Schwartz inequality, and reversing the order of integration, we find that if $k \leq n/2$,

$$\begin{aligned}
\int_{C^*(M)} \|H\xi\|^2 &\leq \int_0^1 \int_M t^{n-2k+2} \left| \int_s^t \|\zeta^{(\tau)}\|_M^2 \, d\tau \right| dt \\
&= \int_0^s \int_M \|\zeta^{(\tau)}\|_M^2 \int_0^\tau t^{n-2k+2} \, dt \, d\tau + \int_s^1 \int_M \|\zeta^{(t)}\|_M^2 \int_\tau^1 t^{n-2k+2} \, dt \, d\tau \\
&\leq \frac{1}{n-2k+3} \left(\int_0^s \int_M \|\zeta^{(\tau)}\|_M^2 \tau^{n-2k+3} \, d\tau + \int_s^1 \int_M \|\zeta^{(\tau)}\|_M^2 \, d\tau \right) \\
&\leq \left(\frac{1+s^{-n+2k-2}}{n-2k+3} \right) \int_{C^*(M)} \|\xi\|^2 < \infty.
\end{aligned}$$

Consequently, $H\xi \in \Omega_{(2)}^{k-1}(C^*(M);\mathbb{R})$; thus, if $\xi \in \Omega_{(2)}^k(M;\mathbb{R})$, and $k \leq n/2$, $H\xi$ is square integrable and

$$\xi = dH\xi + Hd\xi + \pi^*\eta^{(s)}.$$

If $d\xi = 0$,

$$\xi \in d\left(L^2(C^*(M);\Lambda^{k-1})\right) + \pi^* L^2(M;\Lambda^k),$$

which is to say that $\pi^* : H^k(M;\mathbb{R}) \to H_{(2)}^k(C^*(M);\mathbb{R})$ is surjective. Since $d^2 = 0$,

$$d\xi = d(Hd\xi) + d\pi^*\eta^{(s)}$$

Now, if $d\xi \in \pi^* L^2(C^*(M);\Lambda^k)$, then $Hd\xi = 0$ by construction. From this, we have that $\pi^* : H^k(M;\mathbb{R}) \to H_{(2)}^k(M;\mathbb{R})$ is injective for $k \leq n/2$.

All that's left to do is to show $H_{(2)}^k(C^*(M);\mathbb{R}) = 0$ for $k > n/2$. By Cauchy-Schwartz, if $\phi \in \Omega_{(2)}^k(C^*(M);\mathbb{R})$, and $0 < a < b < 1$,

$$\begin{aligned}
\left(\int_b^a \int_M \|\phi^{(t)}\|_M^2 \, dt \right)^2 &\leq \left(\int_0^1 \int_M t^{n-2k} \|\phi^{(t)}\|_M^2 \, dt \right) \left(\int_b^a \int_M t^{2k-n} \, dt \right) \\
&= \left(\int_{C^*(M)} \|\phi\|^2 \right) \left(\int_M 1 \right) \left(\frac{b^{2k-n+1} - a^{2k-n+1}}{2k-n+1} \right).
\end{aligned}$$

Thus, $\int_0^1 \int_M \|\phi^{(t)}\|_M \, dt$ exists if $k \geq n/2$. For almost all $x \in M$, the integral

$$\int_0^t \phi = \int_0^t \phi^{(\tau)} \, d\tau$$

exists for all $t \in (0,1)$. Then, if $\xi = \eta + dt \wedge \zeta \in \Omega_{(2)}^k(C^*(M);\mathbb{R})$, and $k - 1 \geq n/2$, we define

$$H^0\xi = \int_0^t \zeta.$$

Mimicking the argument that showed $H\xi$ was square-integrable, we see that $H^0\xi$ is square-integrable as well, and $\xi = dH^0\xi + H^0 d\xi$. Hence, if ξ is closed, then $\xi = dH^0\xi$, from which it follows $H^k_{(2)}(C^*(M);\mathbb{R}) = 0$ when $k-1 \geq n/2$. We still must worry about the case where n is odd and $k = (n+1)/2$, which is trickier; see Cheeger [C].

With Theorem 3.1 in our toolbelt, we can handle a class of spaces with *isolated metrically conical singularities.*

Definition 3.2. *A compact metric space X of dimension $n+1$ has isolated metrically conical singularities if for some finite set of points $\{p_j\}$ such that $X\setminus \cup_{j=1}^N p_j$ is a smooth Riemannian manifold, and if there exist smooth compact Riemannian n-dimensional manifolds M_j, open neighborhoods U_j of p_j, such that $U_j\setminus\{p_j\}$ is isometric to the cone $C^*_{t_0,t}(M_j) = (t_0,t)\times M_j \subset C^*(M_j)$.*

We say X has *isolated conical singularities* if the metric on $X\setminus \cup_j p_j$ is quasi-isometric to a metric of the form $dt\otimes dt + t^2\pi^* M_j$ in a neighborhood of the p_j. We define the L^2-cohomology of X by

$$H^k_{(2)}(X) = H^k_{(2)}(X\setminus \cup_{j=1}^N p_j).$$

Now, we can generalize even further to pseudomanifolds with conical singularities. Let X be an n-dimensional pseudomanifold– a simplicial complex such that each point is contained in a closed n-simplex, each $(n-1)$-simplex is a face of exactly two n-simplices , and the n-simplices can be compatibly oriented. Alternatively, a Riemannian pseudomanifold X of dimension n is a purely n-dimensional stratified paracompact metric space X which admits a stratification

$$X = X_n \supset \cdots \supset X_1 \supset X_0$$

such that $X_{n-1} = X_{n-2}$ (i.e., $X\setminus X_{n-2}$ is an n-dimensional oriented Riemannian manifold which is dense in X, so that the induced metric on $X\setminus\Sigma$ is quasi-isometric to the flat metric). Let $\Sigma = X_{n-2}$. In such a case, we define

$$H^k_{(2)}(X) = H^k_{(2)}(X\setminus\Sigma).$$

In this case, $\overline{d} = \overline{\delta^*}$, and the Strong Hodge Theorem extends to this case. If X has a stratification by even dimensional strata, then we have the isomorphism $H^k_{(2)}(X) \cong IH_{n-k}(X) \cong \mathrm{Hom}(IH_k(X),\mathbb{C})$, where $IH_k(X)$ denotes the k-th intersection homology group of X ([C], [CGM]).

4. Intersection Homology Theory

We recall the defintion and basic properties of the "middle" intersection homology groups $IH_k(X)$ of Goresky and MacPherson.

Let X be an n-dimensional complex analytic variety, which is contained in a non-singular variety. We can give X a Whitney stratification

$$X = X_n \supset \cdots \supset X_2 \supset X_1 \supset X_0$$

such that

(1) $X_i \backslash X_{i-1}$ is a possibly empty complex analytic i-dimensional manifold,
(2) Whitney's condition B holds with respect to any pair of strata R and S: if $x_j \in S$ is a sequence converging to some $x \in R$, and $y_j \in R$ also converges to x, then the secant lines $\overline{x_j y_j}$ converge to some line ℓ and the tangent spaces $T_{x_j}S$ converge to some limiting plane τ, and $\ell \subset \tau$.

Although we use this stratification to define the groups $IH_k(X)$, the result is independent of the stratification.

Let $C_\bullet(X)$ denote the chain complex of compact (real) subanalytic chains on X, with complex coefficients. The homology of this complex is just the ordinary singular homology $H_*(X;\mathbb{C})$. For any $\xi \in C_i(X)$, let $|\xi|$ denote the support of ξ.

We define the subcomplex $IC_\bullet(X)$ of *allowable chains* by the condition: $\xi \in IC_i(X)$ if

$$\dim |\xi| \cap X_k \leq i - n + k - 1$$

and

$$\dim |\partial\xi| \cap X_k \leq i - n + k - 2.$$

Definition 4.1. *$IH_i(X)$ is defined to be the i-th homology group of the chain complex $IC_\bullet(X)$.*

The groups $IH_i(X)$ satisfy many useful properties, such as having an intersection product $IH_i(X) \times IH_j(X) \to IH_{i+j-2n}$ which leads to a Poincaré duality isomorphism, and behaving nicely with respect to normally nonsingular inclusions and projections (which we will not delve into). Intersection homology (really, its dual, intersection cohomology) also has a axiomatic construction in terms of sheaves (by Deligne), and these objects are of utmost importance in certain categories of perverse sheaves on arbitrary complex analytic spaces.

But, for this paper, we mainly cared about the fact that $IH_\bullet(X)$ is the homology theory dual to L^2-cohomology, at least in the case of spaces with conical singularities. The "local" calculation of $IH_i(X)$ at an isolated singular point is also found to be "dual" to the calculation of Theorem 3.1. The point being, the we have a very concrete topological interpretation of the groups $H^i_{(2)}(X)$, as "**things we integrate over allowable chains**", and the best way to generalize the integration pairing of de Rham cohomology and singular homology to spaces with (reasonable) singularities. To wrap up, if Σ is the singular part of X, and $\alpha \in \Omega^i_{(2)}(X\backslash\Sigma)$, then for almost all allowable chains $\xi \in IC_i(X)$, the integral

$$\int_\xi \alpha$$

exists, and Stokes' theorem holds:

$$\int_\xi d\alpha = \int_{\partial\xi} \alpha.$$

We note that it is never the case that $|\xi| \subset \Sigma$, since

$$\dim |\xi| \cap \Sigma \leq i - n + (n-1) - 1 = i - 2.$$

Thus, we get a natural (integration) pairing

$$\int : H^i_{(2)}(X \backslash \Sigma) \otimes IH_i(X) \to \mathbb{R}.$$

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