Linear Analysis on Manifolds: Notes for Math 7376, Spring 2016

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June 14, 2016

Introduction

These notes were written to accompany a graduate class taught at Northeastern University in Spring 2016. The goal was to cover some classical topics concerning linear elliptic operators on compact manifolds, including elliptic regularity and Fredholm theory, spectral asymptotics (Weyl's formula), and the local Atiyah-Singer index formula using heat kernel methods.

The point of view of these notes is decidedly microlocal, in the sense that operators are studied in terms of their (distributional) Schwartz kernels. Objects of interest, such as generalized inverses of elliptic operators and heat kernels, are first approximated by constructing *parametrices*, which are then improved by some iterative procedure and then compared to the true objects in order to deduce important properties of the latter.

The audience for the class was mixed, with some students having prior expertise with pseudodifferential operators, and other students having limited analytical background. For this reason, the somewhat unusual choice was made to use pseudodifferential operators on manifolds in order to prove key results about elliptic operators, but to skip the technical development of these operators. Thus we take an axiomatic approach, positing the existence of a class of operators satisfying a handful of fundamental axioms, which constitute a kind of *user's interface* for pseudodifferential operators.

There are many good sources for the rigorous development of pseudodifferential operators on manifolds. Among these I mention in particular Pierre Albin's excellent notes [Alb15] written for a similar course at UIUC, which include a detailed background on distribution theory, the requisite Riemannian geometry, and a rather complete development of ΨDOs, in addition to the topics covered here. I followed Albin's approach quite closely in places, and was under the impression that I was complementing his work by covering the Atiyah-Singer theorem in these notes, which was not covered in detail in the first version [Alb12] of notes. It was only after the end of my course that I discovered Albin's later version [Alb15] of his notes, updated to include the index theorem. Thus it is probably the case that these notes constitute a proper subset of Albin's notes, though I hope some readers may yet benefit from the different exposition here, however slightly it may differ!

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Contents

Chapter 1

Elliptic theory on compact manifolds

1.1 Differential operators

Consider $\mathbb{R}^n$ with coordinates $x = (x_1, \ldots, x_n)$. A *differential operator on* $\mathbb{R}^n$ is a linear operator on $C^\infty(\mathbb{R}^n)$ of the form

$$Pu = \sum_{|\alpha| \le k} a_\alpha(x) \partial_x^\alpha u(x). \tag{1.1}$$

Here the *coefficients* $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$ are smooth functions and we we employ *multi-index notation*, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_i \alpha_i$, and ∂_x^α is shorthand for the mixed partial derivative operator

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n}.$$

The integer $k \in \mathbb{N}$ is the *order* of the operator.

Suppose now M is a smooth manifold of dimension n. Recall that this means M has a maximal atlas of smoothly compatible *coordinate charts* (U, U', ϕ), where

$$\phi : U \subset M \overset{\cong}{\to} U' \subset \mathbb{R}^n$$

is a homeomorphism, and "smoothly compatible" means that

$$\phi_b \circ \phi_a^{-1} : U_a' \cap \phi_a(U_b) \subset \mathbb{R}^n \to U_b' \cap \phi_b(U_a) \subset \mathbb{R}^n$$

is a diffeomorphism. We will typically omit ϕ and U' from the notation, and observe the convention of regarding the coordinate functions $x_i := x_i \circ \phi : U \to \mathbb{R}$ as being functions on $U \subset M$ itself; thus we will say $x = (x_1, \ldots, x_n)$ are *local coordinates* on $U \subset M$.

Definition 1.1. A **differential operator of order** k **on** M is a linear operator on $C^\infty(M)$ given locally by expressions of the form (1.1). In other words, on any coordinate chart U, the function Pu restricted to U has the form (1.1).

Observe that this is well-defined; namely, if we have a smooth change of coordinates $x = x(x')$, then $(\partial_{x_1}, \dots, \partial_{x_n}) = \partial_x = D(x, x')^{-1}\partial_{x'}$ where

$$D(x, x') = \left[\frac{\partial x_j}{\partial x'_i}\right]$$

is the Jacobian matrix, whose entries are smooth functions. It follows that (1.1) becomes

$$Pu(x') = \sum_{|\alpha| \leq k} a'_\alpha(x')\partial^\alpha_{x'} u(x')$$

for a new set of coefficients $a'_\alpha(x')$. The general expression for the a'_α in terms of the a_α is quite complicated! However, we will soon see that the top order part behaves nicely.

We denote by $\mathrm{Diff}^k(M)$ the set of differential operators of order at most k on M. It is easy to see that this is a vector space over $\mathbb{R}$ (or $\mathbb{C}$ if we use complex valued functions), and that for all $l \leq k$, we have inclusions

$$\mathrm{Diff}^l(M) \subset \mathrm{Diff}^k(M).$$

In particular $\mathrm{Diff}^0(M) = C^\infty(M)$ is nothing more than the smooth functions on M, considered as multiplication operators on $C^\infty(M)$. $\mathrm{Diff}^1(M)$ includes $\mathrm{Diff}^0(M)$ as well as the smooth *vector fields* $\mathcal{V}(M) = C^\infty(M; TM)$, which we recall is the (vector) space of linear derivations on $C^\infty(M)$:

$$\mathcal{V}(M) \ni V : C^\infty(M) \to C^\infty(M), \quad V(fg) = f\,V(g) + g\,V(f),$$

and these have local coordinate expressions

$$V = a_1(x)\partial_{x_1} + \dots + a_n(x)\partial_{x_n}.$$

Of course, as operators on $C^\infty(M)$, we may *compose* differential operators, and it is easy to see that

$$\mathrm{Diff}^k(M) \circ \mathrm{Diff}^l(M) \subset \mathrm{Diff}^{k+l}(M). \tag{1.2}$$

Again, we may verify this in local coordinates, but observe that if

$$P = \sum_{|\alpha| \leq k} a_\alpha(x)\partial^\alpha_x, \quad Q = \sum_{|\beta| \leq l} b_\beta(x)\partial^\beta_x,$$
$$P \circ Q = \sum_{|\gamma| \leq k+l} c_\gamma(x)\partial^\gamma_x,$$

then the general formulas for c_γ in terms of the a_α and b_β are complicated!

Algebraically speaking, the set

$$\mathrm{Diff}(M) = \bigcup_{k \in \mathbb{N}} \mathrm{Diff}^k(M)$$

of all differential operators has the structure of an associative *filtered algebra*[1], where the filtration is by $\mathbb{N}$. The term 'filtered' here simply reflects the fact that $\mathrm{Diff}(M)$ is a union of subsets indexed by $\mathbb{N}$, and (1.2) holds.

[1] Some prefer the term 'ring' here, which is certainly applicable, though $\mathrm{Diff}(M)$ is also a vector space over $\mathbb{R}$ (or $\mathbb{C}$, if we allow complex coefficients), so we will prefer the term 'algebra'.

1.1.1 Principal symbols

The next order of business is to show that the highest order terms of differential operators behave nicely. Let us revert to the Euclidean setting for just a moment.

Definition 1.2. Let
$$P = \sum_{|\alpha| \le k} a_\alpha(x) \partial_x^\alpha$$
be a differential operator of order k on $\mathbb{R}^n$. The **principal symbol** of P is the (complex-valued) function $\sigma_k(P) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) = C^\infty(T^*\mathbb{R}^n)$ given by
$$\sigma_k(P)(x,\xi) = i^k \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha, \tag{1.3}$$
where $\xi = (\xi_1, \dots, \xi_n)$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Note that the sum is only over terms of order *exactly* k. We will often omit the subscript k from the notation and just write $\sigma(P) := \sigma_k(P)$. Observe that not only is $\sigma(P)$ smooth in both variables, it is in fact a (homogeneous) polynomial of order k in the ξ variables. The factor of i, which is a standard convention, seems a bit annoying at this point, but it would cause much more pain later on if we leave it off. If we are already considering differential operators on complex functions then it is not a big deal, but we are often interested in real functions (or sections of real vector bundles, as below). In this case we can always pass to the complexification to do our analysis, restricting back to the real functions/sections at the end. From this point on, unless otherwise specified, we will usually assume that functions are complex-valued.

Lemma 1.3. *Let P be as above and $f \in C^\infty(\mathbb{R}^n)$. Then*
$$e^{-itf} P e^{itf} = \sum_{j=0}^k t^j P_j,$$
where $P_j \in \mathrm{Diff}^{k-j}(\mathbb{R}^n)$ is independent of t. In particular,
$$P_k = \sigma(P)(x, df_x),$$
where the notation means that if $df_x = \xi_1\, dx_1 + \cdots + \xi_n\, dx_n$, then $\sigma(P)(x, df_x) = \sigma(P)(x,\xi)$.

Proof. Let $df = \sum_j \xi_j(x)\, dx_j$, where $\xi_j = \partial_{x_j} f$. Composing e^{itf} with ∂_{x_j} as operators on smooth functions and taking the commutator gives
$$\partial_{x_j} e^{itf} = e^{itf}(it\,\xi_j + \partial_{x_j})$$
Thus
$$e^{-itf} P e^{itf} = \sum_{|\alpha| \le k} a_\alpha(x)(it\,\xi + \partial_x)^\alpha,$$

where $(it\,\xi+\partial_x)^\alpha = i^{|\alpha|}(t\,\xi_1+\partial_{x_1})^{\alpha_1}\cdots(t\,\xi_n+\partial_{x_n})^{\alpha_n}$. Noting that each $\xi_j=\xi_j(x)$ is a smooth function, and collecting terms of like order in t gives the result. In particular, for the highest order t^k, we get

$$P_k = \sigma(P)(x,df) = i^k \sum_{|\alpha|=k} a_\alpha(x)(df)^\alpha. \qquad \square$$

This result justifies the claim that $\sigma(P)$ may be regarded as a function on $T^*\mathbb{R}^n$, and suggests a coordinate invariant definition which we shall use in the general setting of manifolds.

Definition 1.4. Let $P\in \mathrm{Diff}^k(M)$. The **principal symbol** of P is the smooth function $\sigma(P)\in C^\infty(T^*M)$ (restricting fiberwise to homogeneous polynomials of order k) given by

$$\sigma_k(P)(x,\xi) = \lim_{t\to\infty}(t^{-k}e^{-itf}Pe^{itf})(x),$$

for any $f\in C^\infty(M)$ such that $df_x=\xi$.

Note that, by Lemma 1.3, this depends only on df and not f. One more modification of this will be convenient below. Recall that a homogeneous polynomial on a vector space is uniquely determined by its value on the unit (or any) sphere; indeed, given $p(\omega)$ for $\omega\in\mathbb{S}^{n-1}$, we recover the homogeneous polynomial of order k by $p(t\omega)=t^kp(\omega)$. Alternatively, a homogeneous polynomial determines a section of a trivial line bundle over the **cosphere bundle**

$$S^*M\to M, \quad S^*M = (T^*M\setminus 0)/(0,\infty),$$

where we have written the cosphere bundle as a quotient of the complement of the zero section in T^*M by the dilation action of $(0,\infty)$ given by $\xi\mapsto t\xi$. Technically speaking, these line bundles are not canonically trivialized, but we may choose trivializations (say, by using a Riemannian metric to identify S^*M with the unit sphere bundle in T^*M) and then we may regard the principal symbol as a map

$$\sigma_k : \mathrm{Diff}^k(M)\to C^\infty(S^*M),$$

the advantage being that the maps now have the same image for various k.

Proposition 1.5. *The map $\sigma:\mathrm{Diff}(M)\to C^\infty(S^*M)$ is a homomorphism. That is, if $P\in \mathrm{Diff}^k(M)$ and $Q\in\mathrm{Diff}^l(M)$, then*

$$\sigma_{k+l}(P\circ Q) = \sigma_k(P)\sigma_l(Q).$$

Proof. This is evident from the definition, since

$$\begin{aligned} e^{-itf}PQe^{itf} &= (e^{-itf}Pe^{itf})(e^{-itf}Qe^{itf}) \\ &= (t^k\sigma_k(P)+\mathcal{O}(t^{k-1}))(t^l\sigma_l(Q)+\mathcal{O}(t^{l-1})) \\ &= t^{k+l}\sigma_k(P)\sigma_l(Q)+\mathcal{O}(t^{k+l-1}) \end{aligned}$$

where $\mathcal{O}(t^m)$ denotes terms of order t^l for $l\le m$ times differential operators on M. $\square$

In particular, while differential operators certainly don't commute in general, they *do* commute at the level of principal symbols:

$$\sigma(P \circ Q) = \sigma(P)\sigma(Q) = \sigma(Q)\sigma(P) = \sigma(Q \circ P)$$

even though $P \circ Q \neq Q \circ P$. It is also easy to see that the assignment $P \mapsto \sigma_k(P)$ vanishes precisely on the subset $\mathrm{Diff}^{k-1}(M)$, which is the first part of the following:

Proposition 1.6. *For each k the* symbol sequence

$$\mathrm{Diff}^{k-1}(M) \hookrightarrow \mathrm{Diff}^k(M) \xrightarrow{\sigma} C^\infty(S^*M)$$

is exact.

Exercise 1.1. Prove this. Note that the only part of exactness we have not verified is surjectivity of σ. However this is completely evident in local coordinates, and then different local coordinate expressions may be put together using a partition of unity, where you can ignore any terms involving derivatives of the partition of unity since these will involve differential operators of order $k-1$.

1.1.2 Bundles

We will want to consider not only differential operators acting on smooth functions, but also ones acting between smooth sections of vector bundles.

We recall that a *rank k complex (resp. real) vector bundle* over M is a smooth manifold, E, with a smooth, surjective map $\pi : E \to M$ whose fibers $\pi^{-1}(p)$ have the structure of a vector space over $\mathbb{C}$ (resp. $\mathbb{R}$) of fixed dimension k. Furthermore, there is a covering of M by open sets U along with *local trivializations*, which are diffeomorphisms

$$\phi : \pi^{-1}(U) \overset{\cong}{\to} U \times \mathbb{C}^k$$

with respect to two of which the transition diffeomorphisms $\phi_b \circ \phi_a^{-1}$ on $U_a \cap U_b \times \mathbb{C}^k$ are linear in the second variable. Denote by $C^\infty(M; E)$ the set of smooth *sections* of E, which is to say smooth maps

$$C^\infty(M; E) \ni s : M \to E, \quad \text{s.t.} \quad \pi \circ s = I_M.$$

Let $E \to M$ and $F \to M$ be vector bundles of ranks l and m, respectively. Then a differential operator $P \in \mathrm{Diff}^k(M; E, F)$ is a linear operator from $C^\infty(M; E)$ to $C^\infty(M; F)$ given locally by

$$Ps = \sum_{|\alpha| \leq k} A_\alpha(x) \partial_x^\alpha s(x), \quad A_\alpha \in C^\infty(\mathbb{R}^n; \mathrm{Mat}(m, l, \mathbb{C}))$$

i.e., with coefficients given by local sections of the vector bundle $\mathrm{Hom}(E, F) = E^* \otimes F \to M$.

The definition of the principal symbol given above generalizes to a map

$$\sigma : \mathrm{Diff}(M; E, F) \to C^\infty(S^*M; \pi^* \mathrm{Hom}(E, F)),$$

where $\pi^* \operatorname{Hom}(E, F) = \operatorname{Hom}(\pi^* E, \pi^* F) \to S^*M$ is the pullback of $\operatorname{Hom}(E, F) \to M$ by the cotangent projection $\pi : S^*M \to M$. More pedantically, at a point $(x, \xi) \in S^*M$, $\pi^* \operatorname{Hom}(E, F)$ simply consists of linear maps from E_x to F_x. We will often abuse notation by dropping the π^* and simply writing $\operatorname{Hom}(E, F) \to S^*M$. As in Proposition 1.6, the symbol sequence

$$\operatorname{Diff}^{k-1}(M; E, F) \hookrightarrow \operatorname{Diff}^k(M; E, F) \xrightarrow{\sigma} C^\infty(S^*M; \operatorname{Hom}(E, F))$$

is exact.

We also have composition of the principal symbols in the sense that if $P \in \operatorname{Diff}^k(M; F_1, F_2)$ and $Q \in \operatorname{Diff}^l(M; F_0, F_1)$, then

$$\sigma(P \circ Q) = \sigma(P) \circ \sigma(Q) \in C^\infty\big(S^*M; \operatorname{Hom}(F_0, F_2)\big)$$

with respect to the obvious linear composition map $\operatorname{Hom}(F_1, F_2) \otimes \operatorname{Hom}(F_0, F_1) \to \operatorname{Hom}(F_0, F_2)$ on S^*M.

Remark. We may want to consider real vector bundles E and F, in which case the principal symbol is a section on S^*M of $\pi^* \operatorname{Hom}_{\mathbb{C}}(E, F) = \pi^* \operatorname{Hom}(E, F) \otimes \mathbb{C}$. Alternatively, we may just pass to the complexifications $E_{\mathbb{C}} = E \otimes \mathbb{C}$ and $F_{\mathbb{C}} = F \otimes \mathbb{C}$ at the outset.

Example 1.7. One main example we shall consider is the exterior derivative operator

$$d : C^\infty(M; \Lambda^k) \to C^\infty(M; \Lambda^{k+1}).$$

Here I am using shorthand notation $\Lambda^k = \bigwedge^k T^*M$ exterior powers of the cotangent bundle; so $C^\infty(M; \Lambda^k)$ is the space of smooth k-forms, which is also sometimes denoted $\Omega^k(M)$. This is clearly a differential operator of order 1; in local coordinates

$$d : \sum_{|I|=k} \alpha_I(x)\, dx_I \mapsto \sum_{j=1}^n \sum_{|I|=k} \partial_{x_j} \alpha_I(x)\, dx_j \wedge dx_I$$

where $I \subset \{1, \ldots, n\}$ and dx_I denotes the product $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ with $I = \{i_1 < \cdots < i_k\}$.

To compute the principal symbol of this operator, we may use Definition 1.4 and the derivation property of d to compute

$$d(e^{tf}\, \alpha) = e^{itf}(itdf \wedge \alpha + d\alpha),$$

which implies that

$$\sigma(d)(x, \xi) = i\xi \wedge \cdot \in \pi^* \operatorname{Hom}_{\mathbb{C}}(\Lambda^k, \Lambda^{k+1}) = \pi^* \operatorname{Hom}(\Lambda^k_{\mathbb{C}}, \Lambda^{k+1}_{\mathbb{C}}). \tag{1.4}$$

It is important to think a little bit about what this means: we are at a point $(x, \xi) \in T^*M$ (or possibly S^*M) and we are defining a linear homomorphism from the vector space $(\Lambda^k_{\mathbb{C}})_x = \bigwedge^k T^*_x M \otimes \mathbb{C}$ to the vector space $(\Lambda^{k+1}_{\mathbb{C}})_x = \bigwedge^{k+1} T^*_x M \otimes \mathbb{C}$. The homomorphism is just given by the exterior power with $i\xi$, which is itself a vector in $T^*_x M \otimes \mathbb{C} = (\Lambda^1_{\mathbb{C}})_x$.

1.1.3 Adjoints

Suppose now M is (oriented and) equipped with a Riemannian metric g. Among other things, this means that there is a well-defined *volume form*

$$\mathrm{dVol}_g \in C^\infty(M; \Lambda^n)$$

which is nonvanishing and defined uniquely by the property that whenever $\{e_1, \ldots, e_n\} \subset T_xM$ is a positively oriented orthonormal basis with respect to g,

$$\mathrm{dVol}_g(e_1, \ldots, e_n) = 1.$$

In local (oriented) coordinates, with $g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$, we have

$$\mathrm{dVol}_g = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n.$$

This determines a well-defined smooth measure on M, which is to say an integration functional on the space $C_c^\infty(M)$ of compactly supported functions by

$$C_c^\infty(M) \ni f \mapsto \int_M f \,\mathrm{dVol}_g.$$

The subscript c in $C_c^\infty(M)$ denotes functions with compact support; recall that the **support** of f is the set

$$\mathrm{supp}(f) = \{p : f(p) \neq 0\}^- \subset M$$

given by the closure of the complement of $f^{-1}(0)$. In particular, if M is compact then $C_c^\infty(M) = C^\infty(M)$.

We obtain a (Hermitian) inner product on (complex-valued) functions by

$$C_c^\infty(M; \mathbb{C}) \times C_c^\infty(M; \mathbb{C}) \mapsto \mathbb{C},$$
$$(f, g) = \int_M f\, \overline{g} \,\mathrm{dVol}_g,$$

and taking the completion with respect to the associated norm $\|f\| = (f, f)^{1/2}$ leads to the L^2 space

$$L^2(M; \mathbb{C}) \ni f \iff \Big(\int_M |f|^2 \,\mathrm{dVol}_g\Big)^{1/2} < \infty,$$

which is a Hilbert space in the usual way.

Suppose $E \to M$ is a vector bundle with a Hermitian inner product $\langle \cdot, \cdot \rangle$. This just means that each fiber E_x has a smoothly varying non-degenerate inner product, so that for $s, t \in C^\infty(M; E)$ we have $\langle s, t \rangle \in C^\infty(M; \mathbb{C})$. We may likewise combine this with the volume form to get a non-degenerate pairing on forms:

$$C_c^\infty(M; E) \times C_c^\infty(M; E) \mapsto \mathbb{C},$$
$$(f, g) = \int_M \langle f, g \rangle \,\mathrm{dVol}_g,$$

the completion of which defines the Hilbert space $L^2(M; E)$.

Definition 1.8. The (formal[2]) **adjoint** of a differential operator $P \in \mathrm{Diff}^k(M)$ is the (unique) operator $P^* \in \mathrm{Diff}^k(M)$ with the property that

$$(Pf, g) = (f, P^* g)\,, \quad \text{for all} \quad f, g \in C_c^\infty(M).$$

A similar expression defines the adjoint $P^* \in \mathrm{Diff}^k(M; F, E)$ of an operator $P \in \mathrm{Diff}^k(M; E, F)$ when E and F are Hermitian vector bundles. Note that P^* maps sections of F to sections of E.

Remark. In general, E and F need not be equipped with Hermitian metrics, and in that case the natural adjoint of $P \in \mathrm{Diff}^k(M; E, F)$ is an operator $P^* \in \mathrm{Diff}^k(M; F^*, E^*)$, acting on sections of the dual bundles. However, we shall mostly be concerned with the Hermitian bundle case, in which case we have canonical identifications $E \cong E^*$ and $F \cong F^*$.

If $P \in \mathrm{Diff}^k(M; E, F)$ is given by the local expression $P = \sum_{|\alpha| \leq k} A_\alpha(x) \partial_x^\alpha$, where $A_\alpha \in C^\infty\big(M; \mathrm{Hom}(E, F)\big)$, then locally

$$P^* = \omega(x)^{-1} \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \partial_x^\alpha A_\alpha^*(x) \omega(x) \tag{1.5}$$

where $\mathrm{dVol}_g = \omega(x) dx_1 \wedge \cdots \wedge dx_n$ is the local expression for the volume form and $A_\alpha^* \in \mathrm{Hom}(F, E)$ is the (pointwise) adjoint of A_α with respect to the Hermitian inner products on E and F. (Again, let us emphasize that the above expression is to be understood as a composition of operators on $C^\infty(M; F)$.)

This can be expressed in the form $P = \sum_{|\alpha| \leq k} B_\alpha \partial_x^\alpha$ by commuting the terms ∂_x^α and A_α^*, but as always the general expression for the B_α in terms of the A_α is complicated. At the principal symbol level, however we have the following:

Proposition 1.9. *Let $P \in \mathrm{Diff}^k(M; E, F)$, with respect to Hermitian vector bundles $E, F \to M$. Then the principal symbol of the dual is the dual of the principal symbol:*

$$\sigma(P^*)(x, \xi) = \sigma(P)(x, \xi)^* \in C^\infty(S^*M; \mathrm{Hom}(F, E)).$$

Here the right hand side is the (fiberwise) dual of the section $\sigma(P)$ of $\mathrm{Hom}(E, F)$, giving a section of $\mathrm{Hom}(F, E)$.

Proof. For any $u \in C_c^\infty(M; E)$, $v \in C_c^\infty(M; F)$ and $f \in C^\infty(M)$, we have

$$\begin{aligned} t^k\big(u, \sigma(P^*)(df)v\big) + \mathcal{O}(t^{k-1}) &= \big(u, e^{-itf} P^* e^{itf} v\big) \\ &= \big(e^{-itf} P e^{itf} u, v\big) \\ &= t^k\big(\sigma(P)(df)u, v\big) + \mathcal{O}(t^{k-1}) \\ &= t^k\big(u, \sigma(P)(df)^* v\big) + \mathcal{O}(t^{k-1}), \end{aligned}$$

[2] We use the word ‘formal’ to emphasize that the adjoint is taken with respect to smooth compactly supported functions; it is not (necessarily) the true adjoint of an operator between Hilbert spaces. A proper treatment of the latter leads into technical discussions about domains of unbounded operators, which we will discuss later, but wish to avoid at present.

where we have used that $(e^{itf})^* = e^{-itf}$. It follows that $\sigma(P^*)((x, df_x) = \sigma(P)(x, df_x)^*$ for all $x \in M$, $f \in C^\infty(M)$, and hence $\sigma(P^*)(x, \xi) = \sigma(P)(x, \xi)^*$. □

Exercise 1.2. Prove the formula (1.5), and give an alternate proof of Proposition 1.9 using this local formula, commuting ∂_x^α past $A_\alpha^*(x)$ and $\omega(x)$ and throwing away terms of differential order less than k.

Remark. It is common in Fourier analysis to use the notational convention

$$D_x^\alpha = (-i)^{|\alpha|}\partial_x^\alpha,$$

which is to say we replace the partial derivatives ∂_{x_j} by the operators $D_{x_j} = -i\partial_{x_j}$. Among other reasons (behavior with respect to the Fourier transform being another one), this has the advantage that, on $\mathbb{R}^n$, $(D_x^\alpha)^* = D_x^\alpha$, in contrast to $(\partial_x^\alpha)^* = (-1)^{|\alpha|}\partial_x^\alpha$. Note that, in terms of these operators, the local formula (1.3) for the principal symbol is given by

$$P = \sum_{|\alpha|\leq k} A_\alpha(x) D_x^\alpha \implies \sigma(P)(x, \xi) = \sum_{|\alpha|=k} A_\alpha(x)\xi^\alpha.$$

Example 1.10. Let us compute the adjoint of the operator $d : C^\infty(M; \Lambda^k) \to C^\infty(M; \Lambda^{k+1})$. Recall that a Riemannian metric gives rise to the *Hodge star* operator $\star : \Lambda^k \to \Lambda^{n-k}$, defined by the property that

$$\alpha \wedge \star\beta = \langle \alpha, \beta\rangle \,\mathrm{dVol}_g, \quad \alpha, \beta \in C^\infty(M; \Lambda^k)$$

where $\mathrm{dVol}_g \in C^\infty(M; \Lambda^n)$ is the volume form, and the pairing $\langle \alpha, \beta\rangle$ denotes the Hermitian inner product on Λ^k induced by g. In particular $\mathrm{dVol}_g = \star 1$, and

$$\star(\star\alpha) = (-1)^{k(n-k)}\alpha, \quad \alpha \in C^\infty(M; \Lambda^k). \tag{1.6}$$

Note also that the L^2 pairing on forms is given by $(\alpha, \beta) = \int_M \langle \alpha, \beta\rangle \,\mathrm{dVol}_g = \int_M \alpha \wedge \star\beta$.

Now let $\alpha \in C_c^\infty(M; \Lambda^{k-1})$ and $\beta \in C_c^\infty(M; \Lambda^k)$ and consider the L^2 pairing $(d\alpha, \beta)$. We have

$$\begin{aligned}
(d\alpha, \beta) &= \int_M d\alpha \wedge \star\beta \\
&= \int_M \big(d(\alpha \wedge \star\beta) + (-1)^k \alpha \wedge d(\star\beta)\big) \\
&= \int_M (-1)^{k+(n-k+1)(k-1)} \alpha \wedge \star(\star d \star\beta)
\end{aligned}$$

using (1.6) and Stokes' theorem (since $\alpha \wedge \star\beta$ has compact support, $\int_M d(\alpha \wedge \star\beta) = 0$). Simplifying the signs shows

$$d^* = (-1)^{nk+n+1} \star d \star : C_c^\infty(M; \Lambda^k) \to C_c^\infty(M; \Lambda^{k-1}).$$

It is common convention to denote this operator as

$$\delta := d^* = (-1)^{n(k+1)+1} \star d \star. \tag{1.7}$$

The principal symbol is given by the adjoint of $\sigma(d)(x,\xi) = i\xi \wedge \cdot$, which is the *interior product*

$$\sigma(d^*)(x,\xi) = \sigma(d)(x,\xi)^* = -i\xi^\sharp \lrcorner \cdot \in \mathrm{Hom}_{\mathbb{C}}(\Lambda^k, \Lambda^{k-1}). \tag{1.8}$$

Here $\xi^\sharp \in T_xM$ is the vector dual to $\xi \in T_x^*M$ using the Riemannian product, i.e., the vector defined by $g(\xi^\sharp, v) = \xi(v)$ for all $v \in T_xM$, and $\xi^\sharp \lrcorner \cdot$ is defined by

$$(\xi^\sharp \lrcorner \beta)(v_1, \ldots, v_{k-1}) = \beta(\xi^\sharp, v_1, \ldots, v_{k-1}). \quad \beta \in C^\infty(M, \Lambda^k), \quad v_i \in T_xM,$$

It is easily verified that

$$\xi^\sharp \lrcorner \cdot = (-1)^{nk+n+1} \star (\xi \wedge \star \cdot) \in \mathrm{Hom}_{\mathbb{C}}(\Lambda^k, \Lambda^{k-1}).$$

Example 1.11. The (scalar) **Laplacian** is the operator

$$\Delta = d^*d = - \star d \star d \in \mathrm{Diff}^2(M).$$

By identifying sections of $\Lambda^1 = T^*M$ with sections of TM using the metric, this can be written in the alternate form

$$\Delta f = \mathrm{div}(\nabla f).$$

Indeed, the gradient operator is defined in terms of the operator $\cdot^\sharp : T^*M \to TM$ by $\nabla f = (df)^\sharp$, and the divergence is its formal adjoint. From the symbolic calculus, we may verify that

$$\sigma(\Delta)(x,\xi) = \sigma(d)(x,\xi)^* \sigma(d)(x,\xi) = \xi^\sharp \lrcorner (\xi \wedge \cdot) = |\xi|^2,$$

where the norm of ξ is computed with respect to the Riemannian metric: $|\xi|^2 = g(\xi,\xi)$.

More generally, the Laplacian is defined on forms by

$$\Delta = (d + d^*)^2 = d\,d^* + d^*d \in \mathrm{Diff}^2(M; \Lambda^k),$$

(note that $d^2 = \delta^2 = 0$), and likewise has principal symbol

$$\sigma(\Delta)(x,\xi) = \xi^\sharp \lrcorner (\xi \wedge \cdot) + \xi \wedge (\xi^\sharp \lrcorner \cdot) = |\xi|^2 I \in \mathrm{End}_x(\Lambda^k), \tag{1.9}$$

where I denotes the identity map. The Laplacian is a formally self adjoint operator: $\Delta^* = \Delta$, as follows from the definition.

1.2 Pseudodifferential operators

Having discussed differential operators, we now want to understand their inverses. Of course, differential operators are typically not invertible, but in the case of *elliptic* operators on compact manifolds, the situation is as good as possible: namely, we will show that elliptic operators are invertible up to finite dimensional subspaces of $C^\infty(M)$.

There are two main ways to prove this. The first, functional analytic, approach is to define Sobolev spaces of functions/sections on M and prove direct estimates for elliptic operators with respect to these spaces. The second approach, which we shall pursue here, is to use *pseudodifferential operators*, which are an extension of the differential operators containing the (approximate) inverses of elliptic operators, among other objects.

This second approach is a bit more elegant, but of course nothing comes for free, and there are some necessarily technical details lurking in the proper development and definition of pseudodifferential operators.

We will omit discussion of the construction of pseudodifferential operators on manifolds, and instead we will simply posit their existence and key properties. (See [Alb12] for a development of ΨDOs at a similar level to these notes.) To state these properties, it is a good idea to say a few words about distributions first. (For a comprehensive treatment of distribution theory, see [Hör85].)

1.2.1 Distributions on compact Riemannian manifolds

Definition 1.12. On a compact Riemannian manifold[3] M, the space of **distributions**, denoted $C^{-\infty}(M)$, is the dual space

$$C^{-\infty}(M) := C^{\infty}(M)^* \tag{1.10}$$

which is to say the space of continuous linear functionals $T : C^{\infty}(M) \to \mathbb{C}$. The topology on $C^{-\infty}(M)$ is the weak one, namely $T_j \to T$ in $C^{-\infty}(M)$ if and only if $T_j(\phi) \to T(\phi)$ in $\mathbb{C}$ for all $\phi \in C^{\infty}(M)$.

Recall that $C^{\infty}(M)$ itself is topologized by a family of seminorms

$$\|\phi\|_l = \sup_M \left|\nabla^l \phi\right|$$

where ∇^l denotes an appropriate derivative operator of order l (locally on $\mathbb{R}^n$ we may take $\nabla^l \phi = \sum_{|\alpha| \leq l} \partial_x^{\alpha} \phi$).

Thus $T \in C^{-\infty}(M)$ if there exists a constant $C > 0$ and $k \in \mathbb{N}$ such that

$$T(\phi) \leq C \sum_{l \leq k} \|\phi\|_k\,,$$

and then we say that T has *order* k. In this case T actually defines a linear functional on the Banach space $C^k(M)$, and we may denote by $C^{-k}(M)$ the distributions of order k. (One unfortunate consequence of this notation is the fact that $C^{-0}(M) \neq C^0(M)$; see below.)

There is an injective map $C^{\infty}(M) \hookrightarrow C^{-\infty}(M)$ defined by the L^2 pairing:

$$C^{\infty}(M) \ni u \mapsto T_u \in C^{-\infty}(M), \quad T_u(\phi) = (u, \phi) = \int_M u\overline{\phi}\, \mathrm{dVol}_g, \tag{1.11}$$

[3]If M is not compact, then $C^{-\infty}(M) = C_c^{\infty}(M)^*$ is dual to the space of smooth functions of compact support. If M is not Riemannian, then $C^{-\infty}(M) = C_c^{\infty}(M;\Omega)^*$ is dual to the space of smooth compactly supported *densities*.

and it can be shown that the image of $C^\infty(M)$ is *dense* in $C^{-\infty}(M)$, so that distributions may always be approximated by smooth functions.

The formula (1.11) for T_u is well-defined for any *integrable function* u, so gives an injection

$$L^1(M) \hookrightarrow C^{-\infty}(M), \quad u \mapsto T_u = \int_M u\overline{\phi}\,\mathrm{dVol}_g. \tag{1.12}$$

It is a convenient abuse of notation to write distributions as if they were functions, which is to say we will often write

$$(T,\phi) = \int_M T(x)\overline{\phi}(x)\,\mathrm{dVol}_g(x)$$

for the pairing of a distribution $T \in C^{-\infty}(M)$ and smooth function $\phi \in C^\infty(M)$, even if T is not of the form (1.12) (see below for examples of distributions which are not functions).

Consistent with this convention, and justified rigorously by the density of $C^\infty(M) \subset C^{-\infty}(M)$, we may define various operations on distributions by duality:

Definition 1.13.

(a) (*Multiplication by smooth functions*). If $T \in C^{-\infty}(M)$ and $f \in C^\infty(M)$, the distribution $fT \in C^{-\infty}(M)$ is defined by

$$(fT,\phi) := (T,\overline{f}\phi), \quad \forall\,\phi \in C^\infty(M).$$

(b) (*Differentiation*). If $T \in C^{-\infty}(M)$ and $P \in \mathrm{Diff}(M)$, the distribution $PT \in C^{-\infty}(M)$ is defined by

$$(PT,\phi) := (T,P^*\phi), \quad \forall\,\phi \in C^\infty(M).$$

(c) (*Push-forward*). If $T \in C^{-\infty}(M)$ and $\varphi : M \to N$ is a smooth map of compact Riemannian manifolds, then the push-forward $\varphi_* T \in C^{-\infty}(N)$ is defined by

$$(\varphi_* T,\phi) := (T,\varphi^*\phi) \quad \forall\,\phi \in C^\infty(N).$$

In particular, by part (b), distributions may always be differentiated, and the action of differential operators extends to an action $\mathrm{Diff}(M) : C^{-\infty}(M) \to C^{-\infty}(M)$ such that $\mathrm{Diff}^k(M) : C^{-l}(M) \to C^{-l-k}(M)$.

Definition 1.14. The **support**,

$$\mathrm{supp}(T) \subset M,$$

of $T \in C^{-\infty}(M)$ is the (closure of the) set outside of which it vanishes; more precisely, $p \in M$ is *not* in the support of T if there exists an open neighborhood $U \ni p$ on which T vanishes, i.e., such that $(T,\phi) = 0$ whenever $\mathrm{supp}(\phi) \in U$.

The **singular support**,

$$\mathrm{sing\,supp}(T) \subset M,$$

of $T \in C^{-\infty}(M)$ is the set outside of which the T is smooth; more precisely, $p \in M$ is *not* in the singular support of T if there exists an open neighborhood $U \ni p$ and a smooth $\chi \in C_c^\infty(U)$ such that $\chi T \in C^\infty(M) \subset C^{-\infty}(M)$ coincides with the pairing of a smooth function:

$$(\chi T, \phi) = (u, \phi), \quad \text{for some } u \in C_c^\infty(U).$$

Example 1.15. Note that any (Borel) measure $d\mu$ on M defines a distribution via

$$d\mu : \phi \mapsto \int_M \phi \, d\mu \tag{1.13}$$

whether or not $d\mu$ is absolutely continuous with respect to the Riemannian volume form. In particular, for any $p \in M$, the **Dirac delta distribution** $\delta_p \in C^{-\infty}(M)$ is the distribution defined by the point measure:

$$(\delta_p, \phi) = \phi(p). \tag{1.14}$$

It is easy to see that δ_p cannot be represented by the pairing with any integrable function. Since (1.13) and (1.14) only depend on ϕ and not any of its derivatives, these are distributions of order 0. In fact, by the Riesz representation theorem (the one for measures, not for Hilbert spaces, see [Fol13, Thm. 7.17]), the space $C^{-0}(M)$ of distributions of order 0 is precisely the space of Borel measures on M.

For $P \in \mathrm{Diff}^k(M)$, $k \geq 1$, the distribution $P\,\delta_p$ is an example of a distribution not defined by a measure:

$$(P\delta_p, \phi) = (P^*\phi)(p),$$

and a theorem of Schwartz [Hör85, Thm. 2.3.4] says that any distribution supported at p is of the form $P\delta_p$ for some $P \in \mathrm{Diff}(M)$.

Exercise 1.3. Though it is not a compact manifold, consider some distributions on $\mathbb{R}$. (The crucial difference from a compact manifold is that the test functions ϕ are taken to have compact support.) Show that

$$T(x) = |x|$$

is a locally integrable function, with singular support $\operatorname{sing\,supp}(T) = \{0\}$. Show that

$$T'(x) = \operatorname{sgn}(x) = \begin{cases} 1 & x \geq 0, \\ -1 & x < 0 \end{cases}$$

as distributions (note that it does not matter how you define $\operatorname{sgn}(0)$). Show that

$$T''(X) = \operatorname{sgn}'(x) = 2\delta_0.$$

The distributional sections $C^{-\infty}(M;E)$ of a (Hermitian) vector bundle $E \to M$ may be defined in one of two equivalent ways; either as the space

$$C^{-\infty}(M;E) = C^{-\infty}(M) \otimes C^{\infty}(M;E)$$

of products of smooth sections of E with distributions on M (i.e., as the space of pairs sT, where $s \in C^{\infty}(M;E)$ and $T \in C^{-\infty}(M)$, defined as in Definition 1.13.(a) by $(sT, t) = (T, \langle s, t\rangle)$ for $t \in C^{\infty}(M;E)$), or the dual space

$$C^{-\infty}(M;E) = C^{\infty}(M;E)^*.$$

(Here we are again using the Hermitian structure on E to identify E and E^*; more generally, $C^{-\infty}(M;E)$ should be defined as the dual space $C^{\infty}(M;E^*)^*$.)

The previous definitions (support, singular support, multiplication by smooth functions, action by differential operators) extend in a natural way to $C^{-\infty}(M;E)$, with the exception of the push-forward, which is only well-defined as a map

$$\varphi_* : C^{-\infty}(M;\varphi^*E) \to C^{-\infty}(N;E)$$

from sections of the pullback bundle $\varphi^*E \to M$ to sections of $E \to N$.

The final point to discuss here is the *Schwartz kernel theorem*, which says that any linear operator $A : C^{\infty}(M) \to C^{-\infty}(N)$ from smooth functions on (M,g) to distributions on (N,g') is represented by a distributional *integral kernel* (aka *Schwartz kernel*) $K_A \in C^{-\infty}(M \times N)$:

$$(Au)(x) = \int_M K_A(x,y)u(y)\mathrm{dVol}_g(y).$$

Here again we are abusing notation by writing K_A as a function. The precise statement, taking bundles into account, is the following.

Theorem 1.16 (Schwartz kernel theorem, c.f. [Hör85] Thm. 5.2.1)**.** *Let M and N be a compact Riemannian manifolds with Hermitian vector bundles $E \to M$ and $F \to N$. On the product $M \times N$ let $E \boxtimes F \to M \times N$ and $\mathrm{HOM}(E,F) = E^* \boxtimes F \to M \times N$ be the vector bundles with fibers*

$$E \boxtimes F_{(x,y)} = E_x \otimes F_y \quad \textit{and} \quad \mathrm{HOM}(E,F)_{(x,y)} = \mathrm{Hom}(E_x, F_y) = E_x^* \otimes F_x \quad \textit{respectively.}$$

Then for every linear operator $A : C^{\infty}(M;E) \to C^{-\infty}(N;F)$, there exists a unique distribution $K_A \in C^{-\infty}(M \times N; \mathrm{HOM}(E,F))$ with the property that for every $u \in C^{\infty}(M;E)$ and $v \in C^{\infty}(N;F^*)$,

$$(Au, v) = (K_A, u \boxtimes v)$$

where $u \boxtimes v \in C^{\infty}(N \times M; E \boxtimes F^*)$ is the section given by $(u \boxtimes v)(x,y) = u(x) \otimes v(y)$.

Remark. It is a common abuse of notation to confuse the operator A with its Schwartz kernel, and write

$$Au = \int_M A(x,y)u(y)\,\mathrm{dVol}_g(y).$$

1.2.2 Pseudodifferential operators

We may now discuss the algebra, $\Psi(M)$, of pseudodifferential operators on M, ignoring vector bundles on the first pass.

Proposition 1.17. *Let M be a compact manifold. The set*

$$\Psi(M) = \bigcup_{r\in\mathbb{R}} \Psi^r(M)$$

of pseudodifferential operators *on M, defined by certain Schwartz kernels in $C^{-\infty}(M \times M)$ with singular support contained in the diagonal $\Delta_M = \{(x,x) : x \in M\} \subset M \times M$, has the following properties:*

(a) (Mapping properties). *Each $P \in \Psi^r(M)$ defines an operator*

$$P : C^\infty(M) \to C^\infty(M),$$

and hence by duality

$$P : C^{-\infty}(M) \to C^{-\infty}(M).$$

(b) (Filtered algebra). *If $s \le r$ then $\Psi^s(M) \subseteq \Psi^r(M)$ and $\Psi^s(M) \circ \Psi^r(M) \subset \Psi^{s+r}(M)$, as operators on $C^\infty(M)$. Thus $\Psi(M)$ has the structure of a filtered (over $\mathbb{R}$) algebra.*

(c) (Principal symbols). *For each r, there is a* principal symbol *map*

$$\sigma_r : \Psi^r(M) \to C^\infty(S^*M) \tag{1.15}$$

such that $\sigma_{r+s}(A \circ B) = \sigma_r(A)\sigma_s(B)$ and $\sigma(A^*) = \sigma(A)^*$ (in other words, $\sigma_r : \Psi^r(M) \to C^\infty(S^*M)$ is a $*$-homomorphism), and the symbol sequence

$$\Psi^{r-1}(M) \hookrightarrow \Psi^r(M) \xrightarrow{\sigma} C^\infty(S^*M) \tag{1.16}$$

is exact.

(d) (Extension of differential operators). *For each $k \in \mathbb{N}$, $\mathrm{Diff}^k(M) \subset \Psi^k(M)$, and the principal symbol* (1.15) *extends the one defined for differential operators.*

(e) (Smoothing operators). *The subspace $\Psi^{-\infty}(M) = \bigcap_{r\in\mathbb{R}} \Psi^r(M)$, which is an ideal, is characterized by the operators with smooth Schwartz kernels:*

$$\Psi^{-\infty}(M) = C^\infty(M \times M), \tag{1.17}$$

which is equivalent to the smoothing *property:*

$$A \in \Psi^{-\infty}(M) \iff A : C^{-\infty}(M) \to C^\infty(M). \tag{1.18}$$

(f) (Asymptotic completeness). *For any decreasing, unbounded sequence* (r_j) *in* $\mathbb{R}$*, and sequence of operators* $A_j \in \Psi^{r_j}(M)$*, there exists an operator*

$$A \sim \sum_{j=1}^{\infty} A_j \in \Psi^{r_1}(M), \tag{1.19}$$

which is unique up to terms in $\Psi^{-\infty}(M)$*, where the asymptotic notation* $\sim$ *means that for each* N*,*

$$A - (A_1 + \cdots + A_N) \in \Psi^{r_N - 1}(M). \tag{1.20}$$

Remarks.

- By the Schwartz kernel theorem (Theorem 1.16) it is automatic that $P \in \Psi(M)$ maps smooth functions to distributions; the novelty of part (a) is that Pu is actually smooth if u is. This follows from the fact that the singularities of $P \in C^{-\infty}(M \times M)$ are supported along the diagonal, and are of a special, "conormal" type. A slightly stronger statement, from which (a) follows, is the *pseudolocality* of $\Psi(M)$, which says that

$$\operatorname{sing\,supp}(Pu) \subseteq \operatorname{sing\,supp}(u), \quad \forall\ P \in \Psi(M),\ u \in C^{\infty}(M).$$

- The analogous property of *locality*, that $\operatorname{supp}(Pu) \subseteq \operatorname{supp}(u)$, holds for differential operators but is false in general for pseudodifferential ones. This in turn follows from the characterization of the subset $\operatorname{Diff}(M) \subset \Psi(M)$ of differential operators as those pseudodifferential operators with Schwartz kernels *supported* at the diagonal in $M \times M$ (i.e. actually vanishing outside of the diagonal).

  For example, it is easy to see that the Schwartz kernel of the identity operator, $I \in \operatorname{Diff}^0(M)$, is represented by the Dirac delta distribution of the diagonal, defined by the property

$$u(x) = \int_M \delta(x, y) u(y)\, \mathrm{dVol}_g(y),$$

  for each fixed $x \in M$.

- The smoothing property (1.18) follows from (1.17) and general distribution theory. Essentially, if $P(x, y)$ is smooth in both x and y, then for a fixed distribution $u(y)$, the pairing $\int_M P(x, y) u(y)\, \mathrm{dVol}_g(y)$ makes sense for each $x \in M$ and varies smoothly with x.

- We initially defined the symbol of a differential operator as a function on T^*M (rather than S^*M) which was homogeneous along the fibers. There is a similar definition for pseudodifferential operators, though since smoothness at the zero section and homogeneity of order $r \in \mathbb{R} \setminus \mathbb{N}$ are at odds, it is only required that $\sigma_r(P)(x, \xi)$ be homogeneous for large ξ; thus we may regard

$$\sigma_r(P)(x, \xi) \in C^{\infty}(T^*M), \quad \sigma_r(P)(x, t\xi) = t^r \sigma_r(P)(x, \xi) \quad \forall t,\ |\xi| \geq 1.$$

- In the interest of full disclosure, we should note that in requiring the principal symbol to be homogeneous on T^*M, or equivalently, to be well-defined on S^*M, we are only considering what are sometimes called *classical* pseudodifferential operators. There is in fact a slightly larger algebra of pseudodifferential operators whose symbols are only required to satisfy *symbol estimates* of the form

$$\sigma_r(P) \in S^r(T^*M) \iff \left|\partial_x^\alpha \partial_\xi^\beta \sigma_r(P)(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|^2)^{(r-|\beta|)/2},$$

for all $\alpha, \beta \in \mathbb{N}^n$. In this larger algebra the symbol sequence takes the proper form

$$\Psi^{r-1}(M) \hookrightarrow \Psi^r(M) \xrightarrow{\sigma} S^r(M)/S^{r-1}(M).$$

However, the classical operators (of integer order, in fact) will suffice for the applications we will consider.

The properties (c) and (f) are the keys to doing constructions in the pseudodifferential algebra. We often want to solve some kind of 'algebraic' equation for a pseudodifferential operator Q, for instance the equation $PQ = I$ given a fixed (pseudo)differential operator P, which we will address shortly. The property (c) says that we can consider the associated equation for the principal symbol $\sigma(Q)$, which is vastly simplified by the fact that the symbols are just multiplication operators (and, in the present case, commutative). Provided we can solve the equation for the principal symbol, then the homomorphism and surjectivity properties of σ say that we can then solve the original operator equation modulo $\Psi^{r_0}(M)$ for some finite lower order r_0; in other words we get a Q_0 which solves the equation modulo some error term $R_0 \in \Psi^{r_0}(M)$. We then set out to remove the error term using the principal symbol calculus to obtain an improved solution $Q_0 + Q_1$, which solves the equation modulo an error $R_1 \in \Psi^{r_1}(M)$ for some even lower order r_1, and so on. Proceeding inductively, we develop a series $Q_0 + Q_1 + \cdots$, which can be asymptotically summed by (f) to obtain a solution Q, modulo an error term in the residual ideal $\Psi^{-\infty}(M)$.

Note that this is the best we can do using only the properties above. There is no principal symbol for operators in $\Psi^{-\infty}(M)$, so we cannot really solve away or control these terms, and asymptotic sums as in (f) are only well-defined up to terms in $\Psi^{-\infty}(M)$. To put it another way, we can only use the tools above to solve equations in the quotient algebra $\Psi(M)/\Psi^{-\infty}(M)$. Fortunately, the extremely nice properties (e) of $\Psi^{-\infty}(M)$ mean that this is good enough for many purposes, and in other cases we may be able to use different tools (e.g., from functional analysis) to remove error terms in $\Psi^{-\infty}(M)$.

Before embarking on these ideas in earnest, let us now reintroduce vector bundles into the picture, to obtain the following generalization of Proposition 1.17.

Proposition 1.18. *Let M be a compact manifold, and $E, F \to M$ Hermitian vector bundles. There exists a set*

$$\Psi(M;E,F) = \bigcup_{s\in\mathbb{R}} \Psi^r(M;E,F)$$

defined by certain Schwartz kernels in $C^{-\infty}\big(M\times M; \mathrm{HOM}(E,F)\big)$ with the following properties:

(a) *Each* $P \in \Psi^r(M; E, F)$ *defines operators*

$$P : C^\infty(M; E) \to C^\infty(M; F), \tag{1.21}$$
$$P : C^{-\infty}(M; E) \to C^{-\infty}(M; F). \tag{1.22}$$

(b) *If* $s \le r$ *then* $\Psi^s(M; E, F) \subseteq \Psi^r(M; E, F)$ *and if* $Q \in \Psi^s(M; F, G)$ *and* $P \in \Psi^r(M; E, F)$ *then* $Q \circ P \in \Psi^{s+r}(M; E; G)$.

(c) *For each* $r \in \mathbb{R}$*, there is a principal symbol map*

$$\sigma_r : \Psi^r(M; E, F) \to C^\infty\big(S^*M, \mathrm{Hom}(E, F)\big) \tag{1.23}$$

such that $\sigma_{s+r}(Q \circ P) = \sigma_s(Q) \circ \sigma_r(P) \in C^\infty\big(S^*M; \mathrm{Hom}(E, G)\big)$ *and* $\sigma(P^*) = \sigma(P)^* \in C^\infty\big(S^*M; \mathrm{Hom}(F, E)\big)$*. The symbol sequence*

$$\Psi^{r-1}(M; E, F) \hookrightarrow \Psi^r(M; E, F) \xrightarrow{\sigma} C^\infty\big(S^*M; \mathrm{Hom}(E, F)\big) \tag{1.24}$$

is exact.

(d) *For each* $k \in \mathbb{N}$, $\mathrm{Diff}^k(M; E, F) \subset \Psi^k(M; E, F)$*, and the principal symbol* (1.23) *extends the one defined for differential operators.*

(e) *The subspace* $\Psi^{-\infty}(M; E, F) = \bigcap_{r\in\mathbb{R}} \Psi^r(M; E, F)$ *is characterized by* $\Psi^{-\infty}(M; E, F) = C^\infty\big(M \times M; \mathrm{HOM}(E, F)\big)$ *which is equivalent to the mapping property*

$$R \in \Psi^{-\infty}(M; E, F) \iff R : C^{-\infty}(M; E) \to C^\infty(M; F). \tag{1.25}$$

(f) *For any sequence of operators* $A_j \in \Psi^{r_j}(M; E, F)$ *such that* $r_j \searrow -\infty$*, there exists an operator*

$$A \sim \sum_{j=1}^\infty A_j \in \Psi^{r_1}(M; E, F), \tag{1.26}$$

which is unique up to terms in $\Psi^{-\infty}(M; E, F)$.

1.3 Ellipticity

Definition 1.19. An operator $P \in \Psi^r(M; E, F)$ is **elliptic** if its principal symbol is invertible:

$$P \text{ elliptic} \iff \sigma_r(P) \in C^\infty\big(S^*M; \mathrm{Iso}(E, F)\big) \subset C^\infty\big(S^*M; \mathrm{Hom}(E, F)\big).$$

Remark. If we regard $\sigma_r(P)$ as a section on T^*M rather than S^*M, then the equivalent statement is that $\sigma_r(P)(x, \xi)$ be invertible for all $|\xi| \ge 1$.

Example 1.20. Of the differential operator examples we have considered thus far, $d \in \mathrm{Diff}^1(M; \Lambda^k; \Lambda^{k+1})$ and $\delta \in \mathrm{Diff}^1(M; \Lambda^{k+1}, \Lambda^k)$ are *not* elliptic, as the principal symbols (1.4) and (1.8) are easily seen to have nontrivial kernel for each $\xi \in S_x^* M$.

On the other hand, it follows from (1.9) that the Laplacians $\Delta \in \mathrm{Diff}^2(M; \Lambda^k)$ are elliptic with

$$\sigma(\Delta)^{-1}(x, \xi) = |\xi|^{-1} I \in C^\infty(S^*M; \mathrm{Aut}(\Lambda^k)), \quad \xi \neq 0.$$

Likewise, if we consider the **Hodge-de Rham operator** $d + \delta \in \mathrm{Diff}^1(M; \Lambda)$, where $\Lambda = \bigoplus_{k=0}^n \Lambda^k$ is the total exterior product bundle, then it follows from $(d+\delta)^2 = \Delta \in \mathrm{Diff}^2(M; \Lambda)$ that $d + \delta$ is an elliptic operator of order 1.

1.3.1 Parametrices

We can invert elliptic operators modulo $\Psi^{-\infty}$ using the properties (a)–(f) of Proposition 1.18. Such an (approximate) inverse is called a **parametrix**.

Proposition 1.21. *Let $P \in \Psi^r(M; E, F)$ be an elliptic operator. Then there exists a parametrix $Q \in \Psi^{-r}(M; F, E)$, such that*

$$QP - I = R \in \Psi^{-\infty}(M; E), \quad \textit{and} \tag{1.27}$$

$$PQ - I = R' \in \Psi^{-\infty}(M; F). \tag{1.28}$$

In the first equation I denotes the identity operator in $\Psi^0(M; E)$ and in the second equation I denotes the identity operator in $\Psi^0(M; F)$.

An operator Q such that only (1.27) holds is called a *left parametrix*, and an operator such that only (1.28) holds is called a *right parametrix*.

Proof. We will first prove the existence of a left parametrix. By the definition of ellipticity, $\sigma_r(P) \in C^\infty\big(S^*M; \mathrm{Iso}(E, F)\big)$ is invertible, and then by surjectivity of the symbol in (1.24) there exists $Q_0 \in \Psi^{-r}(M; F, E)$ such that

$$\sigma_{-r}(Q_0) = \sigma_r(P)^{-1} \in C^\infty\big(S^*M; \mathrm{Iso}(F, E)\big).$$

By the composition property it follows that $Q_0 P \in \Psi^0(M; E)$, and that

$$\sigma_0(Q_0 P) = \sigma_{-r}(Q_0)\sigma_r(P) = I \in C^\infty\big(S^*M; \mathrm{End}(E)\big).$$

Note that it follows from the homomorphism property of the principal symbol map that the principal symbol of the identity operator $I \in \Psi^0(M; E)$ is the identity: $\sigma_0(I) = I \in C^\infty\big(M; \mathrm{End}(E)\big)$; in particular, the principal symbols of $Q_0 P \in \Psi^0(M; E)$ and $I \in \Psi^0(M; E)$ agree, so by exactness of (1.24) it follows that

$$Q_0 P - I = R_0, \quad \text{for some } R_0 \in \Psi^{-1}(M; E).$$

Proceeding by induction, suppose that we have obtained operators $Q_i \in \Psi^{-r-i}(M; F, E)$ for $i = 0, \ldots, N$ such that

$$(Q_0 + \cdots + Q_N)P - I = R_N \in \Psi^{-N-1}(M; E), \tag{1.29}$$

and consider the problem of finding $Q_{N+1} \in \Psi^{-r-N-1}(M; E)$ such that $(Q_0 + \cdots + Q_{N+1})P - I \in \Psi^{-N-2}(M; E)$. Expanding out, we find that such a Q_{N+1} must satisfy

$$R_N + Q_{N+1}P = 0 \mod \Psi^{-N-2}(M; E).$$

In particular, it suffices to solve the associated principal symbol equation

$$\sigma_{-N-1}(R_N) + \sigma_{-r-N-1}(Q_{N+1})\sigma_r(P) = 0 \in C^\infty\big(S^*M; \mathrm{End}(E)\big).$$

Using invertibility of $\sigma_r(P)$ again, it follows that we may take Q_{N+1} such that

$$\sigma(Q_{N+1}) = -\sigma(R_N)\sigma(P)^{-1} \in C^\infty\big(S^*M; \mathrm{Hom}(F, E)\big),$$

and then (1.29) holds with N replaced by $N + 1$, and the induction is complete.

Using Proposition 1.18.(f), there exists Q such that

$$Q \sim \sum_{i=0}^{\infty} Q_i \in \Psi^{-r}(M; F, E),$$

and the asymptotic summation property (1.20) and (1.29) together imply that $QP - I \in \Psi^{-N}(M; E)$ for every N, which is to say that (1.27) holds. Note that, by the ideal property of $\Psi^{-\infty}(M; E)$, it follows that another operator $Q' \in \Psi^{-r}(M; F, E)$ is also a left parametrix if and only if $Q - Q' \in \Psi^{-\infty}(M; F, E)$.

A similar procedure can be used to construct a right parametrix $Q' \in \Psi^{-r}(M; F, E)$, such that $PQ' - I = R' \in \Psi^{-\infty}(M; F)$. However, we can use the pseudodifferential properties to show that $Q - Q' \in \Psi^{-\infty}(M; F, E)$. Indeed, consider the composite operator $QPQ' \in \Psi^{-r}(M; F, E)$. It follows that

$$QPQ' = (I - R)Q' = Q' \mod \Psi^{-\infty}(M; F, E),$$

and likewise

$$QPQ' = Q(I - R') = Q \mod \Psi^{-\infty}(M; F, E).$$

Thus $Q - Q' = 0 \mathrm{mod}\ \Psi^{-\infty}(M; F, E)$ and it follows that Q is also a right parametrix (or equivalently, that Q' is also a left parametrix). $\square$

Exercise 1.4. Show that if $P \in \Psi^r(M; E, F)$ is elliptic if and only if $P^* \in \Psi^r(M; F, E)$ is elliptic, and Q is a parametrix for P if and only if Q^* is a parametrix for P^*.

1.3.2 Elliptic regularity

It is from the existence of a parametrix that we deduce the two most important properties of an elliptic operator P. The first says that solutions, or elements of the *distributional nullspace* $\mathrm{Null}(P) = \{u \in C^{-\infty}(M;E) : Pu = 0\}$ are actually smooth, i.e., $\mathrm{Null}(P) \subset C^{\infty}(M;E) \subset C^{-\infty}(M;E)$. More generally, we have the following:

Proposition 1.22 (Elliptic regularity)**.** *Let $P \in \Psi^r(M;E,F)$ be elliptic and suppose $u \in C^{-\infty}(M;E)$ satisfies*

$$Pu = f \in C^{\infty}(M;F). \tag{1.30}$$

Then it follows that u is actually smooth: $u \in C^{\infty}(M;E)$.

Proof. Let $Q \in \Psi^{-r}(M;F,E)$ be a parametrix as in Proposition 1.21. Using (1.27) and (1.30), we have

$$Qf = QPu = (I+R)u = u + Ru \implies u = Qf - Ru.$$

The mapping properties (1.21) and (1.25) imply that both Qf and Ru are actually smooth, hence u is smooth. □

Remark. An equivalent statement is that, while pseudodifferential operators satisfy the general pseudolocality property $\operatorname{sing\,supp}(Pu) \subseteq \operatorname{sing\,supp}(u)$, elliptic operators satisfy the stronger property that $\operatorname{sing\,supp}(Pu) = \operatorname{sing\,supp}(u)$.

1.3.3 Fredholm property of elliptic operators

The second, and perhaps most important, property of elliptic operators on compact manifolds is that they are *Fredholm*, which essentially means invertible off of finite dimensional spaces. This is usually stated as a property of operators between Hilbert spaces such as $L^2(M)$, but since pseudodifferential operators of positive order (hence all interesting differential operators) are unbounded on $L^2(M)$, proceeding entirely via this route would require us to first define Sobolev spaces on M. While we will eventually want to do so, one of the advantages of using the pseudodifferential theory is that we can define and prove the Fredholm property of elliptic operators acting directly on smooth sections (with a small excursion through bounded operators on $L^2(M)$).

Definition 1.23. Let M be a compact Riemannian manifold with $E, F \to M$ Hermitian vector bundles.

A bounded operator $A : L^2(M;E) \to L^2(M;F)$ (or more generally between any pair of Hilbert spaces) is **Fredholm** if

(i) $\mathrm{Null}(A)$ and $\mathrm{Ran}(A)^{\perp} = \{v \in L^2(M;F) : (Au, v) = 0 \ \forall\ u \in L^2(M;E)\}$ are finite dimensional, and

(ii) $\mathrm{Ran}(A)$ is closed, hence there is an orthogonal direct sum decomposition

$$L^2(M;F) = \mathrm{Ran}(A) \oplus \mathrm{Ran}(A)^{\perp}. \tag{1.31}$$

We say an operator $A : C^\infty(M; E) \to C^\infty(M; F)$ is **Fredholm** if

(i) $\mathrm{Null}(A) \subset C^\infty(M; E)$ and

$$\mathrm{Ran}(A)^\perp = \{v : (Au, v) = 0,\ \forall u \in C^\infty(M; E)\} \subset C^\infty(M; F)$$

are finite dimensional, and

(ii) There is a direct sum decomposition

$$C^\infty(M; F) = \mathrm{Ran}(A) \oplus \mathrm{Ran}(A)^\perp \tag{1.32}$$

which is orthogonal with respect to the L^2 pairing.

Note that in a Hilbert space, the statement that $\mathrm{Ran}(A)$ is closed and (1.31) are equivalent, whereas (1.32) *implies* that $\mathrm{Ran}(A)$ is closed but not necessarily vice versa, so we require (1.32) as part of the definition. This definition of Fredholmness on smooth function spaces is not standard.

The point is that solvability for a Fredholm operator is quite close to the finite dimensional situation. Namely, the equation $Au = v$ is solvable if and only if v is orthogonal to the finite dimensional space $\mathrm{Ran}(A)^\perp$ (which is a finite number of conditions), and then the solutions are all of the form $u = u_0 + u_1$ for u_0 any particular solution and u_1 some element of the finite dimensional nullspace $\mathrm{Null}(A)$.

Exercise 1.5. Prove that, in either the L^2 case or the C^∞ case, $\mathrm{Ran}(A)^\perp = \mathrm{Null}(A^*)$, where A^* is the true adjoint in the L^2 case, or the (formal) adjoint with respect to the L^2 pairing in the smooth case.

Exercise 1.6. Prove that A is Fredholm as an operator on L^2 if and only if A^* is Fredholm. As we have currently defined it, this is not necessarily true for a Fredholm operator on smooth sections.

Our strategy is to first prove that an operator of the form $A = I + R$, $R \in \Psi^{-\infty}(M; E)$ is Fredholm in either sense, and then show that this implies that any elliptic pseudodifferential operator is Fredholm in the smooth sense.

Consider the inclusions

$$C^\infty(M; E) \hookrightarrow L^2(M; E) \hookrightarrow C^{-\infty}(M; E)$$

and let $R \in \Psi^{-\infty}(M; E)$. Since R maps the rightmost space to the leftmost one, we may regard it as a bounded operator

$$R : L^2(M; E) \to L^2(M; E). \tag{1.33}$$

We will make use of the following key result.

Lemma 1.24 (Compactness of smoothing operators)**.** *For any $R \in \Psi^{-\infty}(M;E)$ as above, the extension* (1.33) *is a compact operator, meaning the image of any bounded set is precompact. Equivalently, if $\{u_j\}$ is a bounded sequence in $L^2(M;E)$, then $\{Ru_j\}$ has a convergent subsequence.*

Proof. Recall the Arzela-Ascoli theorem ([Fol13, Thm. 4.43]), which says that on a compact space M, any subset of $C^0(M)$ which is bounded and *equicontinuous* is precompact, i.e., a bounded sequence (u_j) for which, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d(x,x') < \delta \implies \left|u_j(x) - u_j(x')\right| < \varepsilon \quad \forall j,$$

(the key point being that δ can be chosen uniformly for all j) has a convergent subsequence. In particular, since uniform bounds on derivatives imply equicontinuity, the inclusion $I : C^1(M) \hookrightarrow C^0(M)$ is compact whenever M is a compact manifold.

Since $R : L^2(M;E)$ to $L^2(M;E)$ factors through the continuous inclusions $C^1(M;E) \subset C^0(M;E) \subset L^2(M;E)$ it follows that (1.33) is compact. □

Lemma 1.25. *Let $R \in \Psi^{-\infty}(M;E)$. Then $A = I - R$ is a bounded operator on $L^2(M;E)$ and on $C^\infty(M;E)$, and is Fredholm in either sense.*

Proof. First consider $\mathrm{Null}(A)$, and note that $Au = (I-R)u = 0$ if and only if $u = Ru$, so as in our elliptic regularity result it follows that $\mathrm{Null}(A) = \left\{u \in L^2 : Ru = 0\right\} = \{u \in C^\infty : Ru = 0\}$. Let $B = \{u \in \mathrm{Null}(A) : \|u\|_{L^2} \leq 1\}$. It follows from Lemma 1.24 that $B = R(B) \subset L^2(M;E)$ is compact (it is closed and precompact). But any subspace of $L^2(M;E)$ with a compact unit ball is finite dimensional; indeed, an infinite orthonormal basis $\{e_i : i \in \mathbb{N}\}$ would have to have a convergent subsequence, which is impossible. We conclude that $\mathrm{Null}(A)$ is finite dimensional.

Next we show that $\mathrm{Ran}(A) \subset L^2(M;E)$ is closed, so suppose u_j is a sequence in $L^2(M;E)$ with $v_j := Au_j \to v$ in L^2. We want to show that $v = Au$ for some $u \in L^2(M;E)$. By the finite dimensionality of $\mathrm{Null}(A)$, we can suppose without loss of generality that $u_j \in \mathrm{Null}(P)^\perp$ for each j. Since $A = I - R$, we have

$$u_j = v_j + Ru_j. \tag{1.34}$$

Consider first the case that $\{u_j : j \in \mathbb{N}\}$ is bounded in $L^2(M;E)$. By Lemma 1.24, (Ru_j) has a convergent subsequence in $L^2(M;E)$, and since v_j converges, it follows from (1.34) that u_j has a convergent subsequence; passing to this subsequence we may assume $u_j \to u \in L^2(M;E)$. It follows by continuity that $Au = v$, so $v \in \mathrm{Ran}(A)$.

Now suppose that $\{u_j\}$ is unbounded in L^2, in particular, after passing to a subsequence, we may assume $\|u_j\|_{L^2} \to \infty$. Then the normalized sequence $u_j' = \frac{u_j}{\|u_j\|}$ in $\mathrm{Null}(A)^\perp$ is bounded, and arguing as above, we conclude that u_j' has a convergent subsequence, say $u_j' \to u' \in L^2(M;E)$. On the other hand, from the fact that $Au_j' = \|u_j\|^{-1} v_j \to 0$, hence $u' \in \mathrm{Null}(A)$ it follows that

$$u' \in \mathrm{Null}(A) \cap \mathrm{Null}(A)^\perp = \{0\},$$

which contradicts the fact that $\|u'\|_{L^2} = \|u_j'\|_{L^2} = 1$. We conclude that $\mathrm{Ran}(A)$ is closed.

Finally, $\operatorname{Ran}(A)^\perp = \operatorname{Null}(A^*)$, where $A^* = I - R^*$ with $R^* \in \Psi^{-\infty}(M;E)$, so the same argument as above shows that this space is finite dimensional. We conclude that A is Fredholm on L^2.

To see that A is Fredholm on C^∞, note that $\operatorname{Null}(A)$ and $\operatorname{Ran}(A)^\perp = \operatorname{Null}(A^*)$ are the same finite dimensional spaces as in the L^2 case by regularity, so it suffices to prove the decomposition (1.32). Let $v \in C^\infty(M;E)$ and consider the (unique) L^2 orthogonal decomposition

$$v = Au_0 + v_1, \quad u_0 \in L^2(M;E), \quad v_1 \in \operatorname{Ran}(A)^\perp = \operatorname{Null}(A^*). \tag{1.35}$$

Since $v_1 \in \operatorname{Null}(A^*)$ is actually smooth, it follows that $Au_0 \in C^\infty(M;E)$, and then that $u_0 = Au_0 + Ru_0 \in C^\infty(M;E)$ by the smoothing property of R. □

Proposition 1.26. *Let $P \in \Psi^r(M;E,F)$ be an elliptic operator. Then $P : C^\infty(M;E) \to C^\infty(M;F)$ is Fredholm.*

Proof. Let Q be a parametrix for P, with

$$QP = I - R, \quad R \in \Psi^{-\infty}(M;E), \tag{1.36}$$
$$PQ = I - S, \quad S \in \Psi^{-\infty}(M;F). \tag{1.37}$$

It follows from (1.36) that $\operatorname{Null}(P) \subseteq \operatorname{Null}(QP) = \operatorname{Null}(I-R)$, which is finite dimensional by Lemma 1.25, hence $\operatorname{Null}(P)$ is finite dimensional. Likewise, from (1.37), it follows that $\operatorname{Ran}(P) \supseteq \operatorname{Ran}(PQ) = \operatorname{Ran}(I-S)$, so $\operatorname{Ran}(P)^\perp \subseteq \operatorname{Ran}(I-S)^\perp$ is a subspace of a finite dimensional space, hence finite dimensional.

To prove the orthogonal direct sum decomposition $C^\infty(M;F) = \operatorname{Ran}(P) \oplus \operatorname{Ran}(P)^\perp$, let $v \in C^\infty(M;F)$ and let

$$v = v_0 + v_1, \quad v_0 \in \operatorname{Ran}(I-S), \quad v_1 \in \operatorname{Ran}(I-S)^\perp,$$

be the smooth orthogonal decomposition afforded by the smooth Fredholm property of $I - S$. Now v_1 is in a finite dimensional space, and so there is a unique orthogonal decomposition

$$v_1 = v_0' + v_1', \quad v_0' \in \operatorname{Ran}(P) \cap \operatorname{Ran}(I-S)^\perp, \quad v_1' \in \operatorname{Ran}(P)^\perp \cap \operatorname{Ran}(I-S)^\perp.$$

Since $\operatorname{Ran}(I-S) \subseteq \operatorname{Ran}(P)$, it follows that

$$v = (v_0 + v_0') + v_1', \quad (v_0 + v_0') \in \operatorname{Ran}(P), \quad v_1' \in \operatorname{Ran}(P)^\perp$$

is a unique orthogonal decomposition. □

One consequence of the Fredholm property is the existence of a **generalized inverse**. Namely, if $A : C^\infty(M;E) \to C^\infty(M;F)$ is Fredholm as in Definition 1.23, Then we may define $G : C^\infty(M;F) \to C^\infty(M;E)$ by

$$Gv = \begin{cases} u : Au = v, \ u \in \operatorname{Null}(A)^\perp, & v \in \operatorname{Ran}(A), \\ 0, & v \in \operatorname{Null}(A^*) = \operatorname{Ran}(A)^\perp. \end{cases}$$

Note the requirement that $u \in \mathrm{Null}(A)^\perp$ means such a u is unique, so G is well-defined, and

$$AG = I - \Pi_{\mathrm{Null}(A^*)}, \quad GA = I - \Pi_{\mathrm{Null}(A)}$$

where $\Pi_{\mathrm{Null}(A)}$ is the orthogonal projection from $C^\infty(M;E)$ onto $\mathrm{Null}(A)$, which can be written as

$$\Pi_{\mathrm{Null}(A)} u = \sum_{j=1}^N (u_j, u)_{L^2}\, u_j$$

for an orthonormal basis $\{u_j : j = 1, \dots, N\}$ of $\mathrm{Null}(A)$, and similarly for $\Pi_{\mathrm{Null}(A^*)}$. In particular, the Schwartz kernel

$$\Pi_{\mathrm{Null}(A)}(x,y) = \sum_{j=1}^N u_j(x)\, u_j^*(y) \in C^\infty\big(M \times M; \mathrm{HOM}(E,E)\big)$$

is smooth, hence the projections $\Pi_{\mathrm{Null}(A)}$, $\Pi_{\mathrm{Null}(A^*)}$ are in $\Psi^{-\infty}(M;E)$ and $\Psi^{-\infty}(M;F)$, respectively.

Proposition 1.27. *Let $P \in \Psi^r(M;E,F)$ be an elliptic pseudodifferential operator. Then the generalized inverse of P is a pseudodifferential operator. In other words there exists a parametrix*

$$\begin{gathered} G \in \Psi^{-r}(M;F,E) : C^\infty(M;F) \to C^\infty(M;E), \quad \textit{s.t.} \\ GP = I - \Pi_{\mathrm{Null}(P)}, \quad PG = I - \Pi_{\mathrm{Null}(P^*)}. \end{gathered}$$

Remark. One way to read this result is to say that, among the various parametrices for P, there is a *best* one, given by the generalized inverse.

Proof. Let $Q \in \Psi^{-r}(M;F,E)$ be any pseudodifferential parametrix, with $QP - I = R \in \Psi^{-\infty}(M;E)$ and $PQ - I = R' \in \Psi^{-\infty}(M;F)$. Writing the operators GPQ and QPG in two different ways gives

$$\begin{aligned} Q - \Pi_{\mathrm{Null}(P)} Q &= GPQ = G + GR', \\ Q - Q\Pi_{\mathrm{Null}(P^*)} &= QPG = G + RG. \end{aligned}$$

From the first equation we can write $G = Q - \Pi_{\mathrm{Null}(P)}Q - GR'$ and plugging this into the term RG in the second equation gives

$$\begin{aligned} G &= Q - Q\Pi_{\mathrm{Null}(P^*)} - R(Q - \Pi_{\mathrm{Null}(P)}Q - GR') \\ &= Q - Q\Pi_{\mathrm{Null}(P^*)} - RQ + R\Pi_{\mathrm{Null}(P)}Q - RGR'. \end{aligned}$$

All of these terms are evidently pseudodifferential except the last one, but this has the property that

$$RGR' : C^{-\infty}(M;F) \to C^\infty(M;E),$$

and this is equivalent to $RGR' \in \Psi^{-\infty}(M;F,E)$ by Proposition 1.18.(e). Thus in fact

$$\begin{aligned} G &= Q - S \in \Psi^{-r}(M;F,E), \\ S &= Q\Pi_{\mathrm{Null}(P^*)} + RQ - R\Pi_{\mathrm{Null}(P)}Q + RGR' \in \Psi^{-\infty}(M;F,E). \end{aligned}$$

$\square$

With the generalized inverse at hand, we can also make a "Fredholm" statement about an elliptic operator acting on distributions:

Proposition 1.28. *Let $P \in \Psi^r(M; E, F)$ be elliptic. Then there are decompositions*

$$C^{-\infty}(M; E) = \mathrm{Ran}(P^*) \oplus \mathrm{Null}(P), \quad C^{-\infty}(M; F) = \mathrm{Ran}(P) \oplus \mathrm{Null}(P^*),$$

in the sense that every distribution $u \in C^{-\infty}(M; E)$ has a unique decomposition $u = u_0 + u_1$ with $u_0 \in \mathrm{Null}(P)$ and $u_1 \in P^*\big(C^{-\infty}(M; F)\big)$, and similarly for $v \in C^{-\infty}(M; F)$. The subspaces $\mathrm{Null}(P)$ and $\mathrm{Null}(P^*)$ are finite dimensional and spanned by smooth sections, and P is an isomorphism from $\mathrm{Ran}(P^*)$ to $\mathrm{Ran}(P)$, with inverse $G \in \Psi^{-r}(M; F, E)$ as constructed above.

Proof. The decompositions come from the operator equations[4]

$$\begin{aligned} I &= P^*G^* + \Pi_{\mathrm{Null}(P)} = GP + \Pi_{\mathrm{Null}(P)} : C^{-\infty}(M; E) \to C^{-\infty}(M; E), \quad \text{and} \\ I &= PG + \Pi_{\mathrm{Null}(P^*)} = G^*P^* + \Pi_{\mathrm{Null}(P^*)} : C^{-\infty}(M; F) \to C^{-\infty}(M; F), \end{aligned}$$

which continue to hold on distributions by continuity from smooth functions. Thus $u \in C^{-\infty}(M; E)$ can be written

$$u = u_0 + u_1 := \Pi_{\mathrm{Null}(P)} u + P^*G^*u$$

and likewise for $v \in C^{-\infty}(M; F)$.

To see that the decomposition is unique, suppose $u = u_0' + u_1'$, with $Pu_0' = 0$ and $u_1' = P^*v_1$ for some v_1. Let $\phi \in C^\infty(M; E)$ be an arbitrary smooth test function and note

$$\begin{aligned} (u, \phi) &= (u_0' + u_1', \phi) \\ &= (u_0' + u_1', P^*G^*\phi + \Pi_{\mathrm{Null}(P)}\phi) \\ &= \underbrace{(u_0', P^*G^*\phi)}_{=0} + (u_0', \Pi_{\mathrm{Null}(P)}\phi) + (u_1', GP\phi) + \underbrace{(u_1', \Pi_{\mathrm{Null}(P)}\phi)}_{=0} \end{aligned}$$

It follows that

$$\begin{aligned} (u_0', \phi) &= (u_0', \Pi_{\mathrm{Null}(P)}\phi) = (u, \Pi_{\mathrm{Null}(P)}\phi) = (\Pi_{\mathrm{Null}(P)}u, \phi), \quad \text{and} \\ (u_1', \phi) &= (u_1', GP\phi) = (u, GP\phi) = (P^*G^*u, \phi), \end{aligned}$$

whence $u_0' = \Pi_{\mathrm{Null}(P)}u$ and $u_1' = P^*G^*u$. □

1.4 Hodge Theory

We can now apply our results on elliptic operators to prove the celebrated Hodge theorem for de Rham cohomology. Later on we will discuss the generalization to arbitrary elliptic complexes.

[4] Note that $P^*G^* = GP$ since both equal $I - \Pi_{\mathrm{Null}(P)}$, and similarly $PG = G^*P^*$.

Let M be a compact Riemannian manifold and for notational convenience throughout this section, denote by

$$\Omega^k(M) = C^\infty(M; \Lambda^k)$$

the space of smooth k-forms. Recall that the *de Rham complex* is the infinite dimensional chain complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M)$$

with *de Rham cohomology spaces*

$$H^k_{\mathrm{dR}}(M; \mathbb{R}) = H^k(\Omega^\bullet, d) = \mathrm{Null}(d : \Omega^k \to \Omega^{k+1}) / \mathrm{Ran}(d : \Omega^{k-1} \to \Omega^k), \quad k = 0, \dots, n.$$

Recall that a form u such that $du = 0$ is said to be *closed*, while if $u = dv$ then u is said to be *exact*. Thus $H^k_{\mathrm{dR}}(M; \mathbb{R})$ is the quotient of the space of closed k-forms by the exact k-forms.

With respect to the Riemannian structure on M, we have, for each degree k, the adjoint operator

$$\delta = d^* = (-1)^{n(k+1)+1} \star d \star : \Omega^k(M) \to \Omega^{k-1}(M)$$

as introduced in Example 1.10, and the Laplacian

$$\Delta = (d + \delta)^2 = d\delta + \delta d : \Omega^k(M) \to \Omega^k(M)$$

as introduced in Example 1.11.

Definition 1.29. A form $u \in \Omega^k(M)$ is **coclosed** if $\delta u = 0$, and **harmonic** if $\Delta u = 0$. The subspace of harmonic k forms is denoted

$$\mathscr{H}^k(M) := \mathrm{Null}(\Delta) \subset \Omega^k(M).$$

Since $\Delta \in \mathrm{Diff}^2(M; \Lambda^k)$ is an elliptic operator, it follows from Proposition 1.26 that each $\mathscr{H}^k(M)$ is finite dimensional. The main result of Hodge theory says that $\mathscr{H}^k(M) \cong H^k_{\mathrm{dR}}(M; \mathbb{R})$, for each k, or equivalently, that each cohomology class has a unique harmonic representative.

Lemma 1.30. *A form is harmonic if and only if it is both closed and coclosed:*

$$\Delta u = 0 \iff du = 0, \ \delta u = 0.$$

Proof. The 'if' direction is clear. For the converse, suppose that $\Delta u = 0$. Then

$$0 = (\Delta u, u) = (d\delta u, u) + (\delta d u, u) = \|\delta u\|^2_{L^2} + \|du\|^2_{L^2},$$

from which it follows that $du = 0$ and $\delta u = 0$. $\square$

Lemma 1.31. *Within $\Omega^k(M)$, the spaces $\mathscr{H}^k(M)$, $\mathrm{Ran}(d) = d(\Omega^{k-1})$ and $\mathrm{Ran}(\delta) = \delta(\Omega^{k+1})$ are pairwise orthogonal. Furthermore, d is injective on $\delta(\Omega^{k+1})$ and δ is injective on $d(\Omega^{k-1})$.*

Proof. Let $u_0 \in \mathscr{H}^k(M)$, $v \in \Omega^{k-1}(M)$ and $w \in \Omega^{k+1}$. Then

$$(u_0, dv) = (\delta u_0, v) = 0, \quad (u_0, \delta w) = (du_0, w) = 0, \quad \text{and} \quad (dv, \delta w) = (d^2 v, w) = 0,$$

which establishes the first claim. For the second, note that $d(\delta w) = 0$ implies

$$0 = (d\delta w, w) = \|\delta w\|^2 \implies \delta w = 0,$$

and similarly $\delta(dv) = 0$ implies

$$0 = (\delta dv, v) = \|dv\|^2 \implies dv = 0. \qquad \square$$

Proposition 1.32 (Hodge decomposition)**.** *For each k, there is an orthogonal decomposition*

$$\Omega^k(M) = \mathscr{H}^k(M) \oplus d(\Omega^{k-1}) \oplus \delta(\Omega^{k+1}),$$

i.e., each $u \in \Omega^k(M)$ has a unique expression $u = u_0 + u_1 + u_2$ with $u_0 \in \mathscr{H}^k(M)$, $u_1 = dv$ and $u_2 = \delta w$.

Remark. Note that v and w are not necessarily unique, but the terms in $u = u_0 + dv + \delta w$ are unique.

Proof. The Laplacian Δ is an elliptic operator which is self-adjoint—in particular $\mathrm{Null}(\Delta^*) = \mathrm{Null}(\Delta) = \mathscr{H}^*(M)$—so from Proposition 1.26 it follows that

$$\Omega^k(M) = \mathscr{H}^k(M) \oplus \Delta(\Omega^k).$$

From the definition of Δ, it follows that $\Delta(\Omega^k) = d\delta(\Omega^k) + \delta d(\Omega^k)$, meaning each term in the former space can be written as a sum of terms in the latter two, but from Lemma 1.31 these latter two spaces are independent and orthogonal, thus

$$\Omega^k(M) = \mathscr{H}^k(M) \oplus d\delta(\Omega^k) \oplus \delta d(\Omega^k). \tag{1.38}$$

Finally, consider the range $d(\Omega^{k-1})$. Since this is orthogonal to the first and last summands in (1.38) by Lemma 1.31, we must have $d(\Omega^{k-1}) \subseteq d\delta(\Omega^k)$, and since the opposite inclusion obviously holds, $d(\Omega^{k-1}) = d\delta(\Omega^k)$. Similarly $\delta(\Omega^{k+1}) = \delta d(\Omega^k)$. $\square$

Theorem 1.33. *For each k, there is an isomorphism*

$$\mathscr{H}^k(M) \cong H^k_{\mathrm{dR}}(M;\mathbb{R}), \quad u_0 \mapsto [u_0].$$

In particular the cohomology spaces $H^k_{\mathrm{dR}}(M;\mathbb{R})$ are finite dimensional, and each class has a unique harmonic representative.

Proof. If $u_0 \in \mathscr{H}^k(M)$, then $du_0 = 0$ by Lemma 1.30, so the map $u_0 \mapsto [u_0]$ is well-defined. To show it is injective, suppose $u_0 = dv$ for some v. Since u_0 and dv are orthogonal by Lemma 1.31, it follows that $u_0 = 0$.

To show the map is surjective, consider an arbitrary class $[u] \in H^k_{\mathrm{dR}}(M;\mathbb{R})$ with representative u. By Proposition 1.32 this has an expression

$$u = u_0 + dv + \delta w, \quad u_0 \in \mathscr{H}^k(M).$$

Since u is closed it follows that $du = d\delta w = 0$, and then by Lemma 1.31 we must have $\delta w = 0$, so in fact

$$u = u_0 + dv,$$

and therefore $[u] = [u_0]$ for some $u_0 \in \mathscr{H}^k(M)$. □

Remark. Another way to view the Hodge theorem is as follows: for each k, write

$$\Omega^k(M) = \Omega^k_0(M) \oplus \Omega^k_+(M), \quad \Omega^k_0(M) = \mathscr{H}^k(M), \quad \Omega^k_+(M) = d(\Omega^{k-1}) \oplus \delta(\Omega^{k+1}).$$

It follows easily that these are subcomplexes (i.e., d preserves the splittings). Moreover, $(\Omega^\bullet_0, d)$ is a *trivial* complex ($d = 0$), hence its cohomology spaces are the same as the chain spaces, while $(\Omega^\bullet_+, d)$ is an *exact* complex (the kernel of d on Ω^k_+ is precisely $d(\Omega^{k-1})$, which is equal to the image of d on Ω^{k-1}_+), hence it has vanishing cohomology.

1.4.1 Distributional Hodge theory

We can also use the elliptic theory to get a Hodge theorem for *distributional* de Rham cohomology, which is the cohomology of the complex

$$C^{-\infty}(M;\Lambda^0) \xrightarrow{d} C^{-\infty}(M;\Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} C^{-\infty}(M;\Lambda^n), \tag{1.39}$$

with adjoint complex

$$C^{-\infty}(M;\Lambda^0) \xleftarrow{\delta} C^{-\infty}(M;\Lambda^1) \xleftarrow{\delta} \cdots \xleftarrow{\delta} C^{-\infty}(M;\Lambda^n).$$

Proposition 1.34. *For each k, there is a unique decomposition*

$$C^{-\infty}(M;\Lambda^k) = \mathscr{H}^k(M) \oplus d\big(C^{-\infty}(M;\Lambda^{k-1})\big) \oplus \delta\big(C^{-\infty}(M;\Lambda^{k+1})\big)$$

and $\mathscr{H}^k(M) \ni u_0 \mapsto [u_0]$ defines an isomorphism between $\mathscr{H}^k(M)$ and the degree k cohomology space of the complex (1.39).

Proof. The proof is largely the same. From Proposition 1.28, we obtain the decomposition $C^{-\infty}(M;\Lambda^k) = \mathscr{H}^k(M) \oplus \Delta(C^{-\infty}(M;\Lambda^k))$, with $\mathscr{H}^k(M)$ the space of smooth harmonic k-forms by elliptic regularity. (In particular, these remain equivalent to the space of forms which are simultaneously closed and coclosed.)

Since $\Delta = d\delta + \delta d$, the second factor is contained in $\mathrm{Ran}(d) + \mathrm{Ran}(\delta)$, and while it does not make sense to say these are orthogonal (we cannot pair two such distributions), it is true that they are independent, since if $u \in \mathrm{Ran}(d) \cap \mathrm{Ran}(\delta)$ then u is both closed and coclosed, hence harmonic, hence vanishing.

Thus we can write every $u \in C^{-\infty}(M; \Lambda^k)$ uniquely as $u = u_0 + dv + \delta w$ for a (smooth) harmonic u_0 and distributional $(k-1)$ and $(k+1)$ forms v and w, respectively. Consider now the map sending $u_0 \in \mathscr{H}^k(M)$ to its class, $[u_0]$ in the cohomology of (1.39). The proof of injectivity is the same: if $u_0 = dv$, then $\|u_0\|^2 = (dv, u_0) = (v, du_0) = 0$ (the pairing makes sense since u_0 is smooth). The proof of surjectivity is also basically the same: any form is written uniquely as $u = u_0 + dv + \delta w$, and if $du = 0$ then $d\delta w = 0$, so δw is both closed and coclosed, hence harmonic, hence vanishing. □

Remark. While the isomorphism between the smooth or distributional cohomology spaces and the spaces of harmonic forms depends on the choice of a Riemannian metric, there is a natural map from smooth to distributional cohomology, which is just the obvious inclusion of smooth forms into distributional ones.

Indeed, a smooth closed form is naturally a distributional closed form, and likewise a smooth exact form is distributionally exact, so we have a homomorphism

$$\begin{aligned} H^k_{\mathrm{dR}}(M; \mathbb{R}) &= \left\{ u \in C^\infty(M; \Lambda^k) : du = 0 \right\} / \left\{ dv : v \in C^\infty(M; \Lambda^{k-1}) \right\} \\ &\to \left\{ u \in C^{-\infty}(M; \Lambda^k) : du = 0 \right\} / \left\{ dv : v \in C^{-\infty}(M; \Lambda^{k-1}) \right\} =: (H^k_{\mathrm{dR}})^{-\infty}(M; \mathbb{R}), \end{aligned}$$

depending only on the smooth structure of M, and one of the consequences of the above is that this is an isomorphism. Thus, it does not matter on a compact manifold M whether we compute de Rham cohomology using smooth forms, distributional forms, or something in between, for instance the L^2 complex

$$L^2(M; \Lambda^0) \xrightarrow{d} L^2(M; \Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} L^2(M; \Lambda^n),$$

(To be precise, since d is an unbounded operator on L^2, we should specify the domain of d, such as the maximal domain or the minimal domain, but it turns out not to matter in this case.)

1.4.2 Elliptic complexes

The Hodge theorem actually applies in a more general setting. Let $E_i \to M$, $i = 0, \ldots, N$ be a sequence of Hermitian vector bundles on a compact Riemannian manifold, with a sequence of differential operators

$$D_i \in \mathrm{Diff}^k(M; E_i, E_{i+1}), \quad \text{s.t.} \quad D_{i+1} \circ D_i = 0$$

with fixed degree k.

Definition 1.35. The complex

$$C^\infty(M; E_0) \xrightarrow{D_0} C^\infty(M; E_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{N-1}} C^\infty(M; E_N) \tag{1.40}$$

is called **elliptic** if for each $(x, \xi) \in S^*M$ the principal symbol sequence

$$(E_0)_x \overset{\sigma(D_0)(x,\xi)}{\longrightarrow} (E_1)_x \overset{\sigma(D_1)(x,\xi)}{\longrightarrow} \cdots \overset{\sigma(D_{N-1})(x,\xi)}{\longrightarrow} (E_N)_x$$

is exact.

For an elliptic complex we can form the associated sequence of *generalized Laplacians*

$$\Delta_i = D_i^* D_i + D_{i-1} D_{i-1}^* \in \mathrm{Diff}^{2k}(M; E_i). \tag{1.41}$$

Exercise 1.7. Show that if the complex is elliptic then the operators (1.41) are elliptic. (Hint: The principal symbol of Δ_i is $\sigma(D_i)^*\sigma(D_i) + \sigma(D_{i-1})\sigma(D_{i-1}^*)$, which at each (x, ξ) is a linear operator on a finite dimensional space. Proceed as in the proofs of Lemmas 1.30 and 1.31 to show that this linear map is injective, hence invertible.)

The analogues of Lemmas 1.30 and 1.31 and Proposition 1.32 hold in this context, with identical proofs, which lead to the following general result, whose proof is identical to that of Theorem 1.33.

Theorem 1.36. *Let* (1.40) *be an elliptic complex. Then for each k there are a isomorphisms*

$$\mathrm{Null}(\Delta_k) \cong H^k\big(C^\infty(M; E_\bullet), D_\bullet\big) \cong H^k\big(C^{-\infty}(M; E_\bullet), D_\bullet\big)$$

between the (necessarily finite dimensional space of) harmonic sections of E_k and the degree k cohomology of (1.40)*, and with the degree k cohomology of the distributional complex associated to* (1.40)*.*

1.5 L^2, Sobolev spaces, and spectral theory

We will next discuss the mapping properties of pseudodifferential operators with respect to L^2 and L^2-based Sobolev spaces (the latter of which we will actually *define* using pseudodifferential operators), and then discuss some basic spectral theory of self-adjoint elliptic operators.

1.5.1 L^2 mapping properties

We have already mentioned boundedness (indeed, compactness) of smoothing operators on L^2. The next step is to extend this to operators of order $r \leq 0$, using a clever trick of Hörmander to reduce to the smoothing case.

Theorem 1.37. *Let $P \in \Psi^0(M; E, F)$. Then P extends to a bounded operator*

$$P : L^2(M; E) \to L^2(M; F),$$

which is compact if $P \in \Psi^r(M; E, F)$ for $r < 0$.

Proof. Hörmander's trick is to show that there exist an operator $Q \in \Psi^0(M;E)$, a smoothing operator $R \in \Psi^{-\infty}(M;E)$ and a constant $c > 0$ such that

$$P^*P + Q^*Q = cI + R. \tag{1.42}$$

To see that (1.42) implies L^2 boundedness, let $u \in C^\infty(M;E)$ and estimate

$$\begin{aligned}\|Pu\|^2_{L^2(M;F)} &\leq \|Pu\|^2_{L^2(M;F)} + \|Qu\|^2_{L^2(M;E)} \\ &= (P^*Pu, u) + (Q^*Qu, u) \\ &= c(u,u) + (Ru, u) \\ &\leq (c + \|R\|_{L^2})\, \|u\|^2_{L^2}\end{aligned}$$

where $\|R\|_{L^2}$ is the operator norm of R on $L^2(M;E)$, which is bounded as per the discussion preceding Lemma 1.24.

To construct Q such that (1.42) holds, we proceed inductively using the symbol calculus. In the first step, we want to solve $Q_0^*Q_0 = c\,I - P^*P$ modulo $\Psi^{-1}(M;E)$. First choose $c > 0$ sufficiently positive that

$$c\,I - \sigma_0(P)^*(x,\xi)\sigma_0(P)(x,\xi) \in \mathrm{End}(E_x) \tag{1.43}$$

is a positive self-adjoint operator for each (x,ξ) in S^*M. (This is possible by compactness: the real-valued function

$$S^*M \ni (x,\xi) \mapsto \sup_{e \in E_x,\ \|e\| \leq 1} \big(\sigma_0(P)(x,\xi)^*\sigma_0(P)(x,\xi)e, e\big)$$

achieves a maximum.) Then we may choose $Q_0 \in \Psi^0(M;E)$ such that $\sigma_0(Q_0)$ is a positive square root of (1.43):

$$\sigma(Q_0)^2 = \sigma(Q_0)^*\sigma(Q_0) = c\,I - \sigma(P)^*\sigma(P).$$

Replacing $Q_0 \in \Psi^0(M;E)$ by $\frac{1}{2}(Q_0 + Q_0^*)$ if necessary (which has the same principal symbol as Q_0), we may assume that Q_0 itself is formally self-adjoint.

By induction, suppose we have formally self-adjoint operators $Q_j \in \Psi^{-j}(M;E)$ for $j = 0, \ldots, N$ such that

$$P^*P + \Big(\sum_{j=0}^{N} Q_j\Big)^2 = c\,I - R_N, \quad R_N \in \Psi^{-N-1}(M;E), \tag{1.44}$$

(note that R_N is automatically self-adjoint) and consider the problem of finding Q_{N+1} such that (1.44) holds with N replaced by $N+1$. Expanding out and collecting terms of pseudodifferential order $-N-1$, we see that it suffices to solve

$$\sigma_0(Q_0)\sigma_{-N-1}(Q_{N+1}) + \sigma_{-N-1}(Q_{N+1})\sigma_0(Q_0) = -\sigma_{-N-1}(R_N).$$

with $\sigma_{-N-1}(Q_{N+1})$ self-adjoint. This finite-dimensional linear algebra problem is always solvable; see Lemma 1.38 below. Then replacing Q_{N+1} by $\frac{1}{2}(Q_{N+1}+Q^*_{N+1})$ again if necessary, we have

$$R_{N+1} := cI - P^*P - \Big(\sum_{j=0}^{N+1} Q_j\Big)^2 \in \Psi^{-N-2}(M;E),$$

with Q_j self-adjoint, completing the induction. Taking an asymptotic sum $Q \sim \sum_{j=0}^{\infty} Q_j$ and replacing Q by $\frac{1}{2}(Q+Q^*)$ completes the proof.

To see why P is compact if $r<0$, note that this implies $\sigma_0(P)=0$ by the symbol sequence, so we may take c as small as we like. In other words, the above argument shows that in this case, for every $\varepsilon>0$ there exists $R_\varepsilon \in \Psi^{-\infty}(M;E)$ such that

$$\|Pu\|^2 \le (\varepsilon + \|R_\varepsilon\|)\,\|u\|^2 .$$

Let (u_j) be a bounded sequence; say $\|u_j\| \le M$ for all j. For each $n\in\mathbb{N}$, $\big(R_{1/n}(u_j)\big)$ has a convergent subsequence by compactness of the R_ε; by a diagonalization argument we can replace u_j by a subsequence such that $\big(R_{1/n}(u_j)\big)$ converges for all n. In particular, given any n, it follows that $\big\|R_{1/n}(u_j-u_k)\big\| \le \frac{1}{n}$ for all i, j sufficiently large, and then

$$\|P(u_j-u_k)\|^2 \le \Big(\tfrac{1}{n}\,\|u_j-u_k\| + \big\|R_{1/n}(u_j-u_k)\big\|\Big)\,\|u_j-u_k\| \le \frac{(2M+1)2M}{n},$$

from which it follows that (u_j) is Cauchy and therefore convergent. □

Lemma 1.38. *Let A be a positive self-adjoint matrix. Then the map $\Phi_A : B \mapsto AB+BA$ is an isomorphism from the space of self-adjoint matrices to itself.*

Proof. Recall that A is unitarily conjugate to a positive diagonal matrix: $D=\operatorname{diag}(\lambda_1,\dots,\lambda_k) = S^*AS$ for some S such that $S^*=S^{-1}$, and observe that $\Phi_A(SBS^*) = S\,\Phi_D(B)\,S^*$. Thus we may assume without loss of generality that $A=D$ is diagonal, with $\lambda_j>0$ by positivity. Letting E_{ij} denote the elementary matrix with entry 1 in the ith row and jth column and 0s otherwise, we compute

$$\Phi_D(E_{ij}+E_{ji}) = (\lambda_i+\lambda_j)(E_{ij}+E_{ji}).$$

Since $E_{ij}+E_{ji}$ for $i\le j$ is a basis for the subspace of self-adjoint matrices and $\lambda_i+\lambda_j \ne 0$, the result follows. □

Exercise 1.8 (Schur's Lemma)**.** Show that, if K is a function on $M\times M$ with the property that

$$C_1 = \sup_{y\in M}\int_M |K(x,y)|\,d\operatorname{Vol}_x < \infty, \quad C_2 = \sup_{x\in M}\int_M |K(x,y)|\,d\operatorname{Vol}_y < \infty,$$

then K is the Schwartz kernel of a bounded operator on $L^p(M)$ for all $p\in(1,\infty)$. (Hint: it is sufficient to show that $|(Ku,v)| \le C_1^{1/p}C_2^{1/q}\,\|u\|_p\,\|v\|_q$ for all $u\in L^p(M)$, $v\in L^q(M)$ where

$\frac{1}{p} + \frac{1}{q} = 1$. To do this, write

$$\begin{aligned} |(Ku, v)| &\leq \int_{M\times M} |K(x,y)|\,|u(y)|\,|v(x)| \;\mathrm{dVol}_x\,\mathrm{dVol}_y \\ &= \int_{M\times M} \big(\,|K(x,y)|^{1/p}\,|u(y)|\,\big)\big(\,|K(x,y)|^{1/q}\,|v(x)|\,\big)\,\mathrm{dVol}_x\,\mathrm{dVol}_y \end{aligned}$$

and use Hölder's inequality.)

This gives another proof that $R \in \Psi^{-\infty}(M)$ extends to a bounded operator on L^2, and in fact on any L^p as well. You may generalize the statement and proof to operators acting between sections of vector bundles.

1.5.2 Unbounded operators and closed extensions

Pseudodifferential operators of positive order are never bounded on L^2, so it is necessary to construct closed domains in L^2 on which they are defined.

Recall that an **unbounded operator** $\big(A, \mathcal{D}(A)\big)$ acting between Hilbert (or more generally, Banach) spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is, by definition a linear map $A : \mathcal{D}(A) \subset \mathcal{H}_1 \to \mathcal{H}_2$ defined on a given subspace $\mathcal{D}(A) \subset \mathcal{H}_1$, called the **domain** of A. We will always assume that $\mathcal{D}(A)$ is *dense* in $\mathcal{H}_1$. We say $\big(A', \mathcal{D}(A')\big)$ is an **extension** of $\big(A, \mathcal{D}(A)\big)$ if $\mathcal{D}(A) \subset \mathcal{D}(A')$ and $A = A'$ on $\mathcal{D}(A)$. The basic problem of interest, of course, is to find extensions of $P \in \Psi^r(M; E, F)$ when $\mathcal{H}_1 = L^2(M; E)$ and $\mathcal{H}_2 = L^2(M; F)$, with initial domain $\mathcal{D}(P) = C^\infty(M; E)$.

For various reasons, spectral theory primarily among them, it is important to consider *closed operators* (aka closed extensions). Recall that an operator $\big(A, \mathcal{D}(A)\big)$ is **closed** if its graph $\operatorname{Gr} A = \{(u, Au) \in \mathcal{D}(A) \times \mathcal{H}_2\}$ is a closed subset of $\mathcal{H}_1 \times \mathcal{H}_2$; in other words, $\big(A, \mathcal{D}(A)\big)$ is closed if, whenever u_j is a sequence in $\mathcal{D}(A)$ such that $u_j \to u$ in $\mathcal{H}_1$ and $Au_j \to v$ in $\mathcal{H}_2$, then in fact $u \in \mathcal{D}(A)$ and $v = Au$.

For the unbounded operator $\Psi^r(M; E, F) \ni P : L^2(M; E) \to L^2(M; F)$ defined on the initial domain $C^\infty(M; E)$, (and in more general settings as well) there are two canonical closed domains. The first is the **minimal domain**

$$\mathcal{D}_{\min}(P) = \big\{u : \exists\, \{u_j\} \subset C^\infty(M; E),\ u_j \to u \text{ in } L^2,\ Pu_j \text{ converges in } L^2\big\},$$

obtained by taking the closure of the graph $\operatorname{Gr} P$, on which Pu is defined to be $\lim Pu_j$, which can be seen to be independent of the sequence u_j in $C^\infty(M; E)$. The second is the **maximal domain**

$$\mathcal{D}_{\max}(P) = \big\{u \in L^2(M; E) : Pu \in L^2(M; F)\big\}$$

defined as those L^2 such that Pu (as a distribution[5]) lies in L^2.

It is easy to see that $\mathcal{D}_{\min}(P) \subseteq \mathcal{D}_{\max}(P)$; indeed, if $u_j \to u$ and Pu_j converges in L^2, then $\lim Pu_j$ is indeed given by Pu, computed as a distribution, since L^2 convergence implies

[5]In the general setting of an unbounded operator on abstract Hilbert spaces, $\mathcal{D}_{\max}(A)$ is defined to be adjoint to $\mathcal{D}_{\min}(A^*)$; in the present setting the distributional definition is more convenient.

convergence in distributions. Furthermore, any other closed domain $\mathcal{D}(P)$ lies in between, i.e., satisfies

$$\mathcal{D}_{\min}(P) \subseteq \mathcal{D}(P) \subseteq \mathcal{D}_{\max}(P).$$

In general, the existence of choices of closed domain is a bit of a pain; fortunately for elliptic operators we have the following result.

Proposition 1.39. *If $P \in \Psi^s(M; E, F)$ is an elliptic operator with $s > 0$, then*

$$\mathcal{D}_{\min}(P) = \mathcal{D}_{\max}(P),$$

i.e., P has a unique closed extension as an unbounded operator from $L^2(M; E)$ to $L^2(M; F)$.

Proof. We must show that $\mathcal{D}_{\max}(P) \subset \mathcal{D}_{\min}(P)$. Thus suppose $u \in \mathcal{D}_{\max}(P)$, so $Pu \in L^2(M; E)$. Let $Q \in \Psi^{-s}(M; F, E)$ be a parametrix, and note

$$u = QPu + Ru \in \operatorname{Ran}(Q) + C^\infty(M; E).$$

Since $C^\infty(M; E) \subset \mathcal{D}_{\min}(P)$, it suffices to show that $\operatorname{Ran}(Q) \subset \mathcal{D}_{\min}(P)$. Thus let $u = Qw \in L^2(M; E)$, with $w \in L^2(M; F)$; we must show that there is a sequence of smooth sections u_j such that $u_j \to u$ in L^2 and Pu_j converges to Pu. To see this, let w_j be a sequence of smooth sections converging in $L^2(M; F)$ to w. Since Q and PQ are bounded on L^2 by Theorem 1.37, it follows that

$$C^\infty(M; E) \ni u_j := Qw_j \to Qw, \quad \text{and} \quad Pu_j = PQw_j \to PQw = Pu,$$

which proves that $\operatorname{Ran}(Q) \subseteq \mathcal{D}_{\min}(P)$. $\square$

In fact, as we shall see in the next section, even more is true: the closed domains $\mathcal{D}_{\max}(P)$ and $\mathcal{D}_{\max}(P')$ for two elliptic operators agree precisely if P and P' have the same order s; then $\mathcal{D}_{\max}(P) = \mathcal{D}_{\max}(P') = H^s(M; E)$ is nothing other than the Sobolev space of order s, which is independent of P.

1.5.3 Sobolev spaces

Besides the Sobolev spaces of positive order, which may be viewed as closed domains in L^2, it is convenient to define L^2-based Sobolev spaces of arbitrary real order, which we initially do in a rather expansive way.

Definition 1.40. Let $s \in \mathbb{R}$. The **Sobolev space** $H^s(M; E)$ of order s is the space

$$H^s(M; E) := \left\{u \in C^{-\infty}(M; E) : Au \in L^2(M; E) \ \forall\, A \in \Psi^s(M; E)\right\}, \tag{1.45}$$

consisting of distributions with image in L^2 under *every* operator of order s.

With this definition it is initially not clear what the topology on $H^s(M; E)$ is or how to practically verify that a section lies in $H^s(M; E)$. These issues are addressed by the following result.

Proposition 1.41. *The space* (1.45) *is equivalently characterized by*

$$H^s(M;E) = \{u \in C^{-\infty}(M;E) : Pu \in L^2(M;F)\} \tag{1.46}$$

for any fixed elliptic operator $P \in \Psi^s(M;E,F)$.

Proof. We first consider the case that $F = E$. Then it suffices to show that any $u \in C^{-\infty}(M;E)$ such that $Pu \in L^2(M;E)$ lies in $H^s(M;E)$, since the other inclusion is obvious. Let $Q \in \Psi^{-s}(M;E)$ be a parametrix with $I - QP = R \in \Psi^{-\infty}(M;E)$, and let $A \in \Psi^s(M;E)$ be arbitrary. Then

$$Au = AQPu + ARu.$$

Since $Pu \in L^2(M;E)$ by assumption and $AQ \in \Psi^0(M;E)$, the first term is in $L^2(M;E)$ by Theorem 1.37. In the second term, $Ru \in C^\infty(M;E)$, so $ARu \in C^\infty(M;E) \subset L^2(M;E)$, which completes the argument.

Repeating the proof with $P \in \Psi^s(M;E,F)$ elliptic and $A = P' \in \Psi^s(M;E)$ elliptic, and vice versa shows that (1.46) holds for an arbitrary pair[6] of vector bundles E and F. □

Note that, if $s \geq 0$, $Pu \in L^2(M;F)$ for $P \in \Psi^s(M;E,F)$ elliptic implies $u \in L^2(M;E)$ by $u = QPu + Ru$, so in fact we have

Corollary 1.42. *For* $s \in [0,\infty)$,

$$H^s(M;E) = \mathcal{D}_{\max}(P) \subseteq L^2(M;E)$$

for any fixed elliptic $P \in \Psi^s(M;E,F)$. *In particular* $H^0(M;E) = L^2(M;E)$.

To put a topology on $H^s(M;E)$ it is convenient to construct a family of invertible elliptic operators $\Lambda_s \in \Psi^s(M;E)$. To do this, for each $s > 0$ let $A_s \in \Psi^{s/2}(M;E)$ be a fixed elliptic operator with $\sigma(A) = I$ and set

$$\Lambda_s = A_s^* A_s + I \in \Psi^s(M;E).$$

This operator is formally self-adjoint and injective ($\Lambda_s u = 0$ implies $\|A_s u\|^2 + \|u\|^2 = 0$ hence $u = 0$), so by the Fredholm theory of §1.3.3 Λ_s is invertible on $C^\infty(M;E)$ and $C^{-\infty}(M;E)$; we let

$$\Lambda_{-s} := \Lambda_s^{-1} \in \Psi^{-s}(M;E)$$

be defined by the generalized inverse (which is in fact a true inverse in this case) of Λ_s, and let $\Lambda_0 = I$. It follows that

$$\Lambda_s : H^s(M;E) \to L^2(M;E)$$

is an isomorphism, and then $H^s(M;E)$ obtains the structure of a Hilbert space with respect to the inner product

$$(u,v)_{H^s} := (\Lambda_s u, \Lambda_s v)_{L^2}.$$

It follows from Corollary 1.44 below that this Hilbert space topology is independent of the choice of Λ_s.

[6] Though of course ellipticity forces $\mathrm{Rank}(E) = \mathrm{Rank}(F)$.

Remark. With a little more work (i.e., by showing that (complex) powers of elliptic operators are pseudodifferential, c.f. [Shu87]) , it is possible to choose the Λ_s such that $\Lambda_s \circ \Lambda_t = \Lambda_{s+t}$, though we shall not need to do so.

In addition, the L^2 pairing extends naturally from $C^\infty(M;E)$ to a nondegenerate bilinear pairing

$$H^s(M;E) \times H^{-s}(M;E) \to \mathbb{C},$$
$$(u,v)_{L^2} \equiv (\Lambda_s u, \Lambda_{-s} v)_{L^2},$$

with respect to which we have a natural identification

$$H^{-s}(M;E) \cong H^s(M;E)^*$$

of $H^{-s}(M;E)$ with the topological dual of $H^s(M;E)$. Note that this pairing is different from the pairing $(u,v)_{H^s}$ which identifies the Hilbert space $H^s(M;E)$ as its own dual.

Thus we now have a complete filtration over $\mathbb{R} \cup \{\pm\infty\}$ of the distributions by regularity, given by

$$C^\infty(M;E) \subset H^s(M;E) \subset L^2(M;E) \subset H^{-s}(M;E) \subset C^{-\infty}(M;E).$$

with 'reflection' across the middle represented by duality.

Remark. In fact it is possible to show as well that $C^\infty(M;E) = H^\infty(M;E) := \bigcap_{s\in\mathbb{R}} H^s(M;E)$ (see Exercise 1.10 below) and likewise that $C^{-\infty}(M;E) = H^{-\infty}(M;E) := \bigcup_{s\in\mathbb{R}} H^s(M;E)$.

The fundamental properties of Sobolev spaces and their interaction with (pseudo)differential operators on M is summarized in the following result.

Theorem 1.43.

(a) *If $r > s$, then the natural inclusion $H^r(M;E) \hookrightarrow H^s(M;E)$ is compact.*

(b) *Let $P \in \Psi^t(M;E,F)$. Then P extends to a bounded linear operator*

$$P : H^s(M;E) \to H^{s-t}(M;F) \tag{1.47}$$

for any $s \in \mathbb{R}$.

(c) *If $P \in \Psi^t(M;E,F)$ is elliptic, then:*

(i) *For each $s \in \mathbb{R}$ there exists a constant $C_s > 0$ such that*

$$\|u\|_{H^s} \leq C(\|Pu\|_{H^{s-t}} + \|u\|_{H^{s-t}}). \tag{1.48}$$

(ii) *P is Fredholm as a bounded operator (1.47), i.e., P has finite dimensional nullspace, and closed range with finite dimensional complement.*

Proof. Using the isomorphisms Λ_* with L^2, the natural inclusion in part (a) is equivalent to the map

$$\Lambda_s \Lambda_{-r} : L^2(M; E) \to L^2(M; E)$$

which is compact since $\Lambda_s \Lambda_{-r} \in \Psi^{s-r}(M; E)$ has order $s - r < 0$. Likewise, the map (1.47) is equivalent to the map

$$\Lambda_{s-t} P \Lambda_{-s} : L^2(M; E) \to L^2(M; F)$$

which is bounded as $\Lambda_{s-t} P \Lambda_{-s} \in \Psi^0(M; E, F)$ has order 0, proving (b).

The basic elliptic estimate (1.48) follows from the existence of a parametrix $Q \in \Psi^{-t}(M; F, E)$ such that $R = I - QP \in \Psi^{-\infty}(M; E)$ and the estimate

$$\begin{aligned} \|u\|_{H^s} &\leq \|QPu\|_{H^s} + \|Ru\|_{H^s} \\ &\leq \|Q\| \, \|Pu\|_{H^{s-t}} + \|R\| \, \|u\|_{H^{s-t}} \,, \end{aligned}$$

where $\|Q\|$ and $\|R\|$ are the operator norms of Q and R as operators from H^{s-t} to H^s.

Finally, the Fredholm property (c).(ii) follows from the generalized inverse equations $GP = I - \Pi_{\mathrm{Null}(P)}$, $PG = I - \Pi_{\mathrm{Null}(P^*)}$, or can be proved directly from (1.48) and the compactness of the inclusion in part (a). □

Note that part (c).(i) implies in particular that the norms $\|\cdot\|_{H^s}$ and $\|P\cdot\|_{H^{s-t}} + \|\cdot\|_{H^{s-t}}$ are equivalent on H^s, since $\|Pu\|_{H^{s-t}}$ and $\|u\|_{H^{s-t}}$ are both controlled by $\|u\|_{H^s}$.

In fact, using the generalized inverse, G, for P in the proof rather than an arbitrary parametrix leads to the following result, justifying our earlier claim that the Hilbert space structure on $H^s(M; E)$ was independent of the choice of Λ_s.

Corollary 1.44. *Let $P \in \Psi^t(M; E, F)$ be an elliptic operator. Then on the space $H^s(M; E)$, there is an equivalence of norms*

$$\|\cdot\|_{H^s} \sim \|P\cdot\|_{H^{s-t}} + \left\|\Pi_{\mathrm{Null}(P)}\cdot\right\| .$$

The last term denotes any norm on Null(P)*, which are all equivalent since it is finite dimensional.*

To compare our definition with the more traditional definition of Sobolev spaces on a manifold[7], recall that $H^s(\mathbb{R}^n)$, $s \geq 0$, is defined as the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \, |\hat{u}(\xi)|^2 \, d\xi$$

(with a naturally associated inner product). Here $\hat{u}(\xi)$ denotes the Fourier transform. By the Parseval identity, we can write this in terms of the original function u by

$$\|u\|^2_{H^s(\mathbb{R}^n)} = \left\|(1 + |D|^2)^{s/2} u\right\|^2_{L^2} ,$$

[7]We ignore vector bundles in the following remarks; it is straightforward to extend to the case of sections of a vector bundle.

where $(1+|D|^2)^{s/2}$ is the operator defined by the oscillatory integral kernel[8]

$$(1+|D|^2)^{s/2} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}(1+|\xi|^2)^{s/2}\,d\xi.$$

This is precisely a self-adjoint, invertible pseudodifferential operator on $\mathbb{R}^n$ of order s with principal symbol given by the identity, in other words, $(1+|D|^2)^{s/2}$ represents a choice of $\Lambda_s \in \Psi^s(\mathbb{R}^n)$, to use our earlier notation.

This traditional definition is transferred from $\mathbb{R}^n$ to a manifold by choosing a covering of M by coordinate charts $\{U_i\}$ supporting a partition of unity $\{\phi_i\}$ and completing $C^\infty(M)$ with respect to the norm

$$u \mapsto \sum_i \|\phi_i u\|_{H^s(\mathbb{R}^n)}.$$

However, this is equivalent to the norm $\|\Lambda_s u\|_{L^2(M)}$ where Λ_s is defined by

$$\Lambda_s = \sum_i \phi_i(1+|D|^2)^{s/2}\phi_i \in \Psi^s(M)$$

using the local coordinate charts, and by our results above this results in the same space.

The local characterization of $H^s(M)$ has some advantages, especially when it comes to relating Sobolev regularity to ordinary derivatives. For example, when $s \in \mathbb{N}$, it is straightforward to prove that there exists $c > 0$ such that

$$c^{-1}(1+|\xi|^2)^{s/2} \le \sum_{0\le|\alpha|\le s} \xi^\alpha \le c(1+|\xi|^2)^{s/2}, \tag{1.49}$$

from which it follows that $H^s(\mathbb{R}^n)$ (and therefore $H^s(M)$) is equivalent to the space of L^2 functions u with s (distributional) derivatives in L^2, i.e., such that $\sum_{|\alpha|\le s} D_x^\alpha u \in L^2$ everywhere locally.

Exercise 1.9. Prove (1.49). Hint: if $\xi = (\xi_1,\dots,\xi_n)$, homogenize by introducing a variable ξ_0 and considering the homogeneous functions $(\xi_0^2+|\xi|^2)^{s/2}$ and $\sum_{|\alpha|\le s}\xi_0^{s-|\alpha|}\xi^\alpha$ restricted to the unit sphere in $\mathbb{R}^{n+1}$.

Likewise, it is straightforward to prove the following standard *Sobolev embedding theorem*:

Theorem 1.45. *If $s > k+n/2$, $n = \dim(M)$, then $H^s(M) \subset C^k(M)$. In particular $H^\infty(M) = \bigcap_s H^s(M) = C^\infty(M)$.*

[8] meaning it defines a distribution in $\mathbb{R}^{2n}$ defined by pairing with a smooth compactly supported function $\phi(x,y)$ and performing the integral in x and y before integrating in ξ.

Exercise 1.10. Prove Theorem 1.45. Hint: since the statement is local, it suffices to prove it for compactly supported functions in $\mathbb{R}^n$, where one may estimate

$$\begin{aligned}|u(x)| &= \left|(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\hat{u}(\xi)\,d\xi\right| \\ &\leq (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}\,|\hat{u}(\xi)|\;d\xi \\ &\leq (2\pi)^{-n}\Big(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-s}\,d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}^n}(1+|\xi|^2)^{s}\,|\hat{u}(\xi)|^2\;d\xi\Big)^{1/2},\end{aligned}$$

using Cauchy-Schwartz. The second factor is $\|u\|^2_{H^s(\mathbb{R}^n)}$, and if $2s > n$ then the first factor converges. Taking the supremum over x proves the result for $k = 0$, and higher derivatives may be taken into account using the fact that $D_{x_j} = -i\partial_{x_j}$ is intertwined with ξ_j by the Fourier transform.

A more traditional approach to elliptic theory on compact manifolds proceeds by defining Sobolev spaces on M by localizing to coordinate charts as above, and then proving the sequence of results in Theorem 1.43 for differential operators. The compact inclusion in part (a) is known as *Rellich's Lemma.* The boundedness (1.47) is straightforward to prove directly for differential operators; the crucial difficulty comes in proving the elliptic estimate (1.48). Once this has been done, elliptic regularity follows by bootstrapping ($Pu \in H^{s-t}$, $u \in H^{s-t}$ implies $u \in H^s$), and Fredholmness of P as an operator $H^s \to H^{s-t}$ follows from the compactness of $H^{s-t} \subseteq H^s$ and an argument similar to the proof of Lemma 1.25.

1.5.4 Spectral theory

The next order of business is to discuss the spectral theory of (self-adjoint) elliptic operators.

Recall that the **spectrum** of a general (possibly unbounded, but closed) operator $\big(A, \mathcal{D}(A)\big)$ on a Hilbert space $\mathcal{H}$ is the set $\operatorname{spec}(A) \subseteq \mathbb{C}$ defined by

$$\lambda \notin \operatorname{spec}(A) \iff (A-\lambda I)^{-1} : \mathcal{H} \text{ bounded},$$

i.e., as the complement of the set of λ such that $A - \lambda I : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ admits a bounded inverse. Generally speaking, there are several ways for $A - \lambda I$ to fail to be invertible:

- $A - \lambda I$ may not be injective, i.e., $\operatorname{Null}(A - \lambda I)$ may be non-empty. In this case we say λ is an **eigenvalue**, and the set, $\operatorname{spec}_{\mathrm{pt}}(A)$ of such λ is called the **point spectrum**.
- $A - \lambda I$ may be injective but not be surjective. In this case we further subdivide into the **continuous spectrum**

  $$\operatorname{spec}_{\mathrm{c}}(A) = \{\lambda : (A - \lambda I) \text{ injective with dense range, not surjective.}\}$$

  and the **residual spectrum**

  $$\operatorname{spec}_{\mathrm{res}}(A) = \{\lambda : (A - \lambda I) \text{ injective without dense range.}\}$$

These various types of spectrum, as well as the dependence of $\mathrm{spec}(A)$ on the choice of domain $\mathcal{D}(A)$, make life generally complicated. Fortunately in the case of interest to us, the story is vastly simplified.

As we have seen, an elliptic operator $P \in \Psi^s(M; E)$ for $s > 0$ forms an unbounded operator on $L^2(M; E)$ with the unique closed domain $\mathcal{D}(P) = H^s(M; E)$. If P is formally self-adjoint, meaning $P^* = P$ as an operator on $C^\infty(M; E)$ then by uniqueness of the closed domains for elliptic operators it follows that P is **essentially self-adjoint**, i.e., there is a unique closed domain $\mathcal{D}(P) = \mathcal{D}(P^*)$ on which $P = P^*$.

Let us now make use of ellipticity. Note first that, since $\lambda I \in \Psi^0(M; E)$ has order strictly less than that of P, the operator $P - \lambda I$ is elliptic if and only if P is.

If $\lambda \in \mathrm{spec}_{\mathrm{pt}}(P)$ is an eigenvalue, then by ellipticity of $P - \lambda I$, the eigenspace $\mathrm{Null}(P - \lambda I)$ is finite dimensional. Note that if u_1 and u_2 are eigenfunctions with eigenvalues λ_1 and λ_2, then

$$\lambda_1(u_1, u_2) = (Pu_1, u_2) = (u_1, Pu_2) = \overline{\lambda}_2(u_1, u_2).$$

With $u_2 = u_1$ it follows that any eigenvalues of P must be real, and with $\lambda_1 \neq \lambda_2$ it follows that eigenfunctions with distinct eigenvalues are orthogonal.

Next, by Fredholm theory, $P - \lambda I : H^s(M; E) \to L^2(M; E)$ always has closed range, so P cannot have continuous spectrum, and from $\mathrm{Ran}(P - \lambda I)^\perp = \mathrm{Null}(P^* - \overline{\lambda} I) = \mathrm{Null}(P - \overline{\lambda} I)$, it follows that P cannot have residual spectrum either: if P was not surjective, then the complement of its range would consist of eigenvectors with eigenvalue $\overline{\lambda}$, which is equal to λ by reality.

We conclude that the spectrum of P consists entirely of eigenvalues, and in fact the situation is as nice as possible:

Theorem 1.46. *Let $P \in \Psi^s(M; E)$, $s > 0$ be a self-adjoint elliptic operator. Then* $\mathrm{spec}(P) \subset \mathbb{R} \subset \mathbb{C}$ *forms a discrete set $\{\lambda_j : j \in \mathbb{N}\}$ with $|\lambda_j| \to \infty$, and there is a complete orthonormal basis of $L^2(M; E)$ consisting of eigenvectors of P.*

The proof follows directly from the spectral theorem for self-adjoint compact operators on a Hilbert space, which is a standard result:

Theorem 1.47 (c.f. [Tay96a], Prop. 6.6, Appendix A.)**.** *Let A be a self-adjoint, compact operator on the Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ has a complete orthonormal basis of eigenvectors $\{u_j : Au_j = \lambda_j u_j\}$, and the eigenvalues $\{\lambda_j\} \subset \mathbb{R}$ form a sequence of real numbers with* 0 *as the only possible accumulation point.*

Exercise 1.11. Prove (or look up the proof of) Theorem 1.47, which is relatively simple. Here is an outline:

(1) Any eigenspaces of A are finite dimensional (by compactness) and orthogonal (by self-adjointness), with real eigenvalues (by self-adjointness), and the orthocomplement of an eigenspace is invariant under A (by self-adjointness).

(2) Either $\|A\|$ or $-\|A\|$ is an eigenvalue, by the following steps.

(a) By compactness, $u \mapsto \|Au\|$ achieves a maximum on the unit ball, so there exists u with $\|u\| = 1$ and $\|Au\| = \|A\|$.

(b) For any w orthonormal to u,

$$\|A\|^2 + 2s\,\mathrm{Re}(A^2u, w) + s^2 \|Aw\|^2 = \|A(u + sw)\|^2 \leq \|A\|^2 (1 + s^2)$$

for all $s \in \mathbb{R}$, which as $s \to 0$ implies $(A^2u, w) = 0$, so u is an eigenvalue of A^2 with eigenvalue $\|A\|^2$.

(c) Obtain an eigenvector of A with eigenvalue $\pm \|A\|$ by taking either $v = \|A\| u + Au$, or u itself (if $v = 0$).

(3) By induction, suppose we have eigenvalues λ_i and eigenspaces E_i, $i = 1, \ldots, N$, with $|\lambda_i| > |\lambda_{i+1}|$, such that $A|_{(\bigoplus_{i=1}^N E_i)^\perp}$ has norm $\|A\| < \lambda_N$. Then by the above there is an eigenvalue λ_{N+1} with $|\lambda_{N+1}| = \left\| A|_{(\oplus_i E_i)^\perp} \right\|$, and off of the corresponding eigenspace E_{N+1}, A must have strictly smaller norm.

Proof of Theorem 1.46. We know $\mathrm{spec}(P) \subset \mathbb{R}$, and it can't be all of $\mathbb{R}$, since then orthogonality of eigenvectors would then imply the existence of an uncountable orthonormal set in $L^2(M; E)$, which is a separable Hilbert space[9]. Thus there exists some $\lambda_0 \in \mathbb{R}$ such that $P - \lambda_0 I$ is invertible.

Since $(P - \lambda_0 I)^{-1} \in \Psi^{-s}(M; E)$ is a compact, self-adjoint operator, Theorem 1.47 applies, and then it suffices to note that

$$Pu = \lambda u \iff (P - \lambda_0 I)^{-1} u = \frac{1}{\lambda - \lambda_0} u.$$

(Add and subtract $\lambda_0 u$ on the LHS and multiply by $(P - \lambda_0 I)^{-1}$.) Thus there is a complete orthonormal basis of $L^2(M; E)$ consisting of eigenvectors of P, which form a sequence $\{\lambda_j\} \subset \mathbb{R}$ with $|\lambda_j| \to \infty$, since $\frac{1}{\lambda_j - \lambda_0}$ may only accumulate at 0. □

From this we obtain one of the fundamental results for compact Riemannian manifolds:

Corollary 1.48. *Let (M, g) be a compact manifold. Then the space $L^2(M)$ admits a complete orthonormal basis of eigenfunctions of the scalar Laplacian $\Delta \in \mathrm{Diff}^2(M)$, with eigenvalues (written with multiplicity)*

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \tag{1.50}$$

forming a sequence such that $\lambda_j \to +\infty$.

Proof. It only remains to verify that the eigenvalues are non-negative and contain 0. The first follows from the fact that Δ is a positive operator:

$$(\Delta u, u) = (d^* du, u) = \|du\|^2 \geq 0,$$

and the second follows from the fact that constant functions have eigenvalue 0 (i.e., are harmonic). □

[9]If you don't like this argument, you can take $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ below, but then you have to employ the spectral theorem for *normal* compact operators, rather than self-adjoint ones.

Definition 1.49. We refer to the sequence (1.50) as the **spectrum** of the Riemannian manifold (M, g).

One way to view Corollary 1.48 is that it gives us a kind of *Fourier decomposition* of functions on M. When solving PDE involving the Laplacian, such as the *heat equation* $(\partial_t + \Delta)u = f$, the *Schrödinger equation* $(i\partial_t + \Delta)u = f$ or the *wave equation* $(\partial_t^2 + \Delta)u = f$, we may decompose u and f into eigenfunctions, thereby achieving a kind of separation of variables.

From another point of view, we may consider the association $(M, g) \mapsto (\lambda_j : j \in \mathbb{N})$. Clearly the spectrum is determined entirely by the metric on M; in particular if two Riemannian manifolds are isometric, then they have identical spectra. The converse question, or **inverse problem**, of whether a compact Riemannian manifold is determined up to isometry by its spectrum (to paraphrase Mark Katz's famous question, "Can you hear the shape of a compact Riemannian manifold?") is false in general. Indeed, sporadic examples of isospectral but non-isometric manifolds have been known going back to Milnor in 1964, and there are systematic constructions of such examples going back to Sunada from 1985. However, the question of just how much of the geometry of (M, g) is encoded by the spectrum (1.50) is a topic of ongoing research.

Chapter 2

Spectral theory and heat kernels

2.1 Overview

A general version of the spectral theorem for an unbounded, self-adjoint operator $\big(A, \mathcal{D}(A)\big)$ on a Hilbert space $\mathcal{H}$ says that there exists a **projection-valued measure** $dE(\lambda)$ on $\mathbb{R}$, meaning a Borel measure on $\mathbb{R}$, supported on the spectrum of A, taking values in projection operators on $\mathcal{H}$, such that we have the **spectral resolution**

$$A = \int \lambda \, dE(\lambda).$$

Then, to any function f which is measurable with respect to $dE(\lambda)$, we obtain a new (possibly unbounded) operator

$$f(A) = \int f(\lambda) \, dE(\lambda),$$

with A itself associated to the identity function and I associated to the constant function 1. This association of f to $f(A)$ is known as the **functional calculus**.

If $P \in \Psi^s(M; E)$, $s > 0$ is self-adjoint and elliptic, then the measure is simple to describe. Indeed, from Theorem 1.46, the spectrum consists entirely of discrete eigenvalues $\mathrm{spec}(P) = \{\lambda_j\}$, which we write with multiplicity for convenience, and the measure is atomic, of the form

$$dE(\lambda) = \sum_j \delta(\lambda - \lambda_j) \, \Pi_j,$$

where $\delta(\lambda - \lambda_j)$ is the Dirac measure concentrated at λ_j, and Π_j denotes projection onto the (necessarily 1-dimensional since we count eigenvalues with multiplicity) eigenspace associated with λ_j. Letting $\{e_j \in C^\infty(M; E)\}$ denote an L^2 orthonormal basis of eigenfunctions, the Schwartz kernel of Π_j is simply $e_j(x)e_j^*(y)$ and the spectral resolution is

$$P = \sum_j \lambda_j e_j(x) e_j^*(y), \quad f(P) = \sum_j f(\lambda_j) e_j(x) e_j^*(y).$$

An operator of particular interest is the following.

Definition 2.1. Suppose $P \in \Psi^s(M;E)$, $s > 0$ is self-adjoint and positive. (In particular $\mathrm{spec}(P) \subset \mathbb{R}_+ := [0,\infty)$.) Then the **heat kernel** associated to P is the family of operators

$$e^{-tP} = \int_{\mathbb{R}_+} e^{-t\lambda}\, dE(\lambda) = \sum_j e^{-t\lambda_j} e_j(x) e_j^*(y), \quad t \in \mathbb{R}_+, \tag{2.1}$$

In light of the identities[1]

$$\begin{aligned} (\partial_t + P)e^{-tP} &= 0, \quad t > 0, \\ e^{-tP}|_{t=0} &= I, \end{aligned} \tag{2.2}$$

the heat kernel gives a (forward) fundamental solution to the associated (homogeneous) **heat equation**

$$\begin{aligned} (\partial_t + P)u &= 0 \quad \text{on } \mathbb{R}_+ \times M, \\ u(0,x) &= u_0(x) \in L^2(M;E) \end{aligned} \tag{2.3}$$

in $C^1\big(\mathbb{R}_+; L^2(M;E)\big)$, meaning that $u = e^{-tP}u_0$ is the unique solution to (2.3). Uniqueness of solutions to (2.3) follows from the fact that their L^2 norms are decreasing in t via

$$\partial_t \|u\|_{L^2}^2 = 2(\partial_t u, u) = -2(Pu,u) \le 0,$$

so $u_0 = 0$ implies $u \equiv 0$. This also implies uniqueness of the operator solutions to (2.2).

In addition to the homogeneous initial value problem, the heat kernel can also be used to solve the associated *inhomogeneous* equation

$$\begin{aligned} (\partial_t + P)u &= f \\ u(0,x) &= u_0(x), \end{aligned} \tag{2.4}$$

where $f = f(t,x)$ is a function of t with values in $L^2(M;E)$. Indeed, **Duhamel's formula** says that (2.4) admits the solution

$$u = e^{-tP}u_0 + \int_0^t e^{-(t-s)P} f(s,\cdot)\, ds.$$

(See the discussion of convolution operators in §2.3.7.)

While the spectral theory of self-adjoint operators provides the existence of e^{-tP} on abstract grounds, and (2.1) gives a representation of its Schwartz kernel, it is in general quite difficult to extract useful information from this representation. Below we will construct a parametrix (approximate forward fundamental solution) for the heat equation in the case that P is a second order, Laplace-type differential operator (and not necessarily self-adjoint), which will allow us to determine various properties of the heat kernel.

[1] Technically speaking, the differentiation $\partial_t e^{-tP} = -Pe^{-tP}$ requires justification, and attention must be paid to the domains on which each side is bounded. However, it is straightforward to check that e^{-tP} is a bounded operator for all $t \ge 0$; indeed we shall later show that it is compact (in fact trace-class) for $t > 0$. Likewise, one can show that the differentiation identity is valid on the domain of P.

In fact, rather than using information about the spectrum of P to solve the associated heat equation, we may turn the problem around and use the fundamental solution to the heat equation to get information about the spectrum. To motivate this, consider that we can pass from the projection-valued measure $dE(\lambda) = \sum \delta_{\lambda_j} \Pi_j$ on $\mathbb{R}_+$ to a real-valued measure on $\mathbb{R}_+$ by taking the trace of each projection Π_j (which simply counts the number of independent eigenvectors; since we have counted with multiplicity this number is always 1).

Definition 2.2. For P as above, define the **spectral measure** by

$$d\mu(\lambda) := \operatorname{Tr} dE(\lambda) = \sum \delta_{\lambda_j} \operatorname{Tr} \Pi_j = \sum \delta_{\lambda_j}.$$

This is an atomic measure on $\mathbb{R}_+$, and gives rise to the **eigenvalue counting function**

$$N(l) := |\{\lambda_j \in \operatorname{spec}(P) : \lambda_j \leq l\}| = \int_0^l d\mu(\lambda),$$

which is simply the count of all the eigenvalues (with multiplicity) less than or equal to l.

Notice that, since the trace is a linear functional, it follows that, for any f in the functional calculus for which $\operatorname{Tr} f(P)$ is well-defined, i.e., such that $f(P)$ is **trace-class** (see §2.4), then we expect to have have

$$\int_{\mathbb{R}_+} f(\lambda)\, d\mu(\lambda) = \int_{\mathbb{R}_+} f(\lambda) \operatorname{Tr} dE(\lambda) = \operatorname{Tr} \int f(\lambda) dE(\lambda) = \operatorname{Tr} f(P).$$

In particular, the *Laplace transform*[2] of the counting measure $d\mu(\lambda)$ is the distribution

$$\int e^{-t\lambda}\, d\mu(\lambda),$$

which by the above reasoning, we expect to coincide with the trace of the heat kernel, $\operatorname{Tr} e^{-tP}$, should the latter be well-defined. As one generally expects from harmonic analysis, transforms such as Fourier or Laplace generally encode asymptotic behavior of a distribution as $\lambda \to \infty$ in terms of the asymptotic behavior of the transform as $t \to 0$. This is indeed the case, as we will later see in the form of *Karamata's Tauberian Theorem* (c.f. Prop. 2.24).

2.2 Heat kernel of Laplacian on Euclidean space

The approach we shall follow regarding the construction of heat kernel parametrices is via a manifold with corners called the "heat space", following [Mel93]. This is also the approach followed in [Alb12]. A more traditional approach (originally due to Hadamard) can be found in [BGV92], among other sources.

[2] As a special case of the Fourier transform, the Laplace transform is well-defined, say, on tempered distributions.

In preparation for our parametrix construction let us first consider the heat kernel for the ordinary scalar Laplacian

$$\Delta = -\sum_{i=1}^{n} \partial_{x_i}^2 = \sum_{i=1}^{n} D_{x_i}^2 \in \mathrm{Diff}^2(\mathbb{R}^n)$$

on Euclidean space, which can be explicitly identified. We seek the fundamental solution $H(t, x, y) \in C^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^{2n})$ to the equation

$$(\partial_t + \Delta)H = 0, \quad H(0, x, y) = \delta_0(x - y).$$

By translation invariance of all operators involved, it follows that H has the form $H(t, x, y) = h(t, x - y)$ where $h \in C^{-\infty}(\mathbb{R}_+ \times \mathbb{R}^n)$ satisfies

$$(\partial_t + \Delta)h = 0, \quad h(0, x) = \delta_0(x).$$

Taking the Fourier transform in x, this becomes

$$(\partial_t + |\xi|^2)\widehat{h} = 0, \quad h(0, \xi) = 1,$$

with the unique solution $\widehat{h}(t, \xi) = e^{-t|\xi|^2}$. By a standard calculation, the inverse Fourier transform of this is $h(t, x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$, from which we obtain

$$H(t, x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\Big(- \frac{|x - y|^2}{4t}\Big). \tag{2.5}$$

Exercise 2.1. Prove that if $\hat{f}(\xi) = e^{-t|\xi|^2}$ then $f(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$.

There are several things to remark about (2.5). The first is that $H(t, x, y)$ is a smooth function for $t > 0$. Furthermore, for fixed $x \neq y$, $H(t, x, y)$ vanishes rapidly as $t \to 0$. The only singularity of $H(t, x, y)$ is therefore at the set $\{0\} \times \mathbb{R}^n_{\mathrm{diag}} \subset \mathbb{R}_+ \times \mathbb{R}^{2n}$, i.e., at the diagonal at time 0.

Let us consider the precise manner in which $H(t, x, y)$ is singular here. First of all, there is the overall singular factor of $t^{-n/2}$, though this is comparatively mild. More importantly is the term $\exp\left(-\frac{|x-y|^2}{4t}\right)$. Evidently this admits different limits depending on the path of approach to the singular locus; along a path such that

$$|x - y|^2 = c\, t, \quad c \geq 0,$$

the limit is $\exp\left(-\frac{c}{4}\right)$. To motivate the constructions below, we determine a "change of variables" with respect to which $H(t, x, y)$ becomes smooth.

First of all, the homogeneity of $|x - y|^2/4t$ along with the factor $t^{-n/2}$ strongly suggest introducing

$$\tau := \sqrt{t}$$

as a coordinate. Of course this is not a diffeomorphism, but the map $\tau \mapsto t = \tau^2$ is a smooth map from $\mathbb{R}_+$ to itself. Since the kernel only depends on $x - y$, it is also useful to write $z = (x - y)/2$, and take $w = (x + y)/2$ (or even $w = x$; it doesn't really matter). We now have

$$H(\tau, z, w) = C_n \tau^{-n} \exp\Big(- \Big|\frac{z}{\tau}\Big|^2 \Big), \quad C_n = (4\pi)^{-n/2}.$$

on the product manifold $(\mathbb{R}_+)_\tau \times \mathbb{R}^n_z \times \mathbb{R}^n_w$.

Next, consider the first two factors $(\mathbb{R}_+)_\tau \times \mathbb{R}^n_z$ as half-space of $\mathbb{R}^{n+1}$ and use polar coordinates. Thus write

$$\mathbb{R}^{n+1} \ni (\tau, z) = r\omega,$$
$$\omega \in \mathbb{S}^n_+ := \Big\{ (\omega_0, \omega') \in \mathbb{R}_+ \times \mathbb{R}^n : \omega_0^2 + \left|\omega'\right|^2 = 1 \Big\}.$$

In these new variables, we have

$$H(r, \omega, w) = C_n (r\omega_0)^{-n} \exp\Big(- \Big|\frac{\omega'}{\omega_0}\Big|^2 \Big).$$

Notice that the exponential factor is smooth since $\omega_0^2 + |\omega'|^2 = 1$. Furthermore, as a function on the hemisphere $\mathbb{S}^n_+$, ω_0 is equal to 1 in the center, and tends smoothly to 0 at the boundary, $\partial\mathbb{S}^n_+ = \mathbb{S}^{n-1}$, and the exponential decay of $\exp\left(-1/\omega_0^2\right)$ compensates for the ω_0^{-n} term. We conclude that

$$r^n H(r, \omega, w) \text{ is smooth on } (\mathbb{R}_+)_r \times \mathbb{S}^n_+ \times \mathbb{R}^n.$$

Of course, (r, ω) are not actually coordinates on $\mathbb{R}_+ \times \mathbb{R}^n$ at $r = 0$; rather they define a smooth surjection $\mathbb{R}_+ \times \mathbb{S}^n_+ \to \mathbb{R}_+ \times \mathbb{R}^n$, $(r, \omega) \mapsto r\omega$, which is a diffeomorphism for $r > 0$. This is an example of a **radial blow-up**, which we will consider in more detail in the next section. The composite map

$$\begin{aligned} \beta : \mathbb{R}_+ \times \mathbb{S}^n_+ \times \mathbb{R}^n &\to \mathbb{R}_+ \times \mathbb{R}^{2n}, \\ (r, \omega, w) &\mapsto (t, x, y) = (r^2\omega_0^2, w + r\omega', w - r\omega'), \end{aligned} \tag{2.6}$$

is a smooth surjection from a manifold with corners to a manifold with boundary, which is again a diffeomorphism away from the boundaries. To restate the above, it has the property that

$$\beta^* H \in r^{-n} C^\infty(\mathbb{R}_+ \times \mathbb{S}^n_+ \times \mathbb{R}^n),$$

which is to say it effectively *resolves the singularity* of $H(t, x, y)$ at $t = 0$.

This will be the guiding principle for the heat space construction for a manifold, which we consider next.

2.3 Heat kernel on a manifold

Let M be a compact Riemannian manifold and $P \in \mathrm{Diff}^{\ell}(M;E)$ an elliptic differential operator (not necessarily self-adjoint). Guided by the above analysis of the heat kernel for $\partial_t + \Delta$ on $\mathbb{R}^n$, we will construct a space supporting a parametrix for the heat kernel of $\partial_t + P$ on M. We will proceed generally at first and then specialize to the case that $\ell = 2$ and the principal symbol $\sigma_2(P)$ is a scalar multiple of the identity. However, the construction below can be modified to handle the general case (and even the case that P is pseudodifferential).

2.3.1 Blow-up

Definition 2.3. Let N be a compact manifold (possibly with boundary), and $Y \subset N$ a submanifold (which may lie in ∂N). The **radial blow-up**, $[N;Y]$, of Y in N is the space constructed as follows. As a set,

$$[N;Y] = (N \setminus Y) \cup S_+Y,$$

where S_+Y denotes the *inward pointing spherical normal bundle* to Y:

$$S_+Y = \{[v] \in NY/(0,\infty) : v = \partial_t \chi(t),\ \exists\, \chi : [0,1] \to N\}$$

consisting of normal vectors which are the limits of paths in N through Y (hence inward pointing). We define the (surjective) **blow-down map**

$$\beta : [N;Y] \to N, \quad \beta(p) = \begin{cases} p & p \in N \setminus Y, \\ \pi(p) & p \in S_+Y \end{cases},$$

by the identity away from Y and the bundle projection $\pi : S_+Y \to Y$ over Y. The topology and smooth structure on $[N;S]$ are generated by the functions

$$\left\{ f/g : f, g \in C^\infty(N),\ Y = f^{-1}(0) = g^{-1}(0),\ df|_Y, dg|_Y \neq 0 \right\}$$

of smooth functions vanishing simply and only at Y. By L'Hopital's formula these functions have well-defined limits[3] on open sets of S_+Y. We refer to the boundary face $S_+Y \subset [N;Y]$ as the **front face** of the blow-up.

If N either has no boundary or $Y \cap \partial N = \emptyset$, then $[N;Y]$ is a manifold with boundary (with a new boundary component consisting of the face $S_+Y = SY$). If $Y \subset \partial N$, then $[N;Y]$ is a **manifold with corners**, meaning it has coordinate charts modeled on open sets in $\mathbb{R}^k_+ \times \mathbb{R}^n$ (here with $k \leq 2$). In fact, the radial blow-up may be defined if N is already a manifold with corners and Y is any suitably nice submanifold, though we shall not need this full generality. The blow-down map is a surjective submersion which restricts to a diffeomorphism away from Y.

[3] At $[v]$, take the limit along any path with $\partial_t \chi(t) = v$, noting that the result is independent of χ and v.

To connect this definition with our earlier comments, note that if (locally, say) Y is the inclusion of the origin in $N = \mathbb{R}^n$ or $N = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ then the polar coordinates $(r, \omega) \in \mathbb{R}_+ \times \mathbb{S}^{n-1}_{(+)}$ lift to a system of coordinates on $[N; Y]$ which are nondegenerate all the way down to $r = 0$. Thus radial blow-up can be regarded as the act of "taking polar coordinates seriously" along Y.

2.3.2 Heat space

Returning to the consideration of $\partial_t + P$, $P \in \mathrm{Diff}^\ell(M; E)$, we first introduce the ℓth root of t in the space $\mathbb{R}_+ \times M^2$ via the map

$$\begin{aligned} \varrho : \mathbb{R}_+ \times M^2 &\to \mathbb{R}_+ \times M^2, \\ (\tau, x, y) &\mapsto (\tau^\ell, x, y) = (t, x, y). \end{aligned} \tag{2.7}$$

Having done this, we then blow up $\{0\} \times M_{\mathrm{diag}}$, to obtain the manifold with corners

$$M^2_H := [\mathbb{R}_+ \times M^2; \{0\} \times M_{\mathrm{diag}}] \tag{2.8}$$

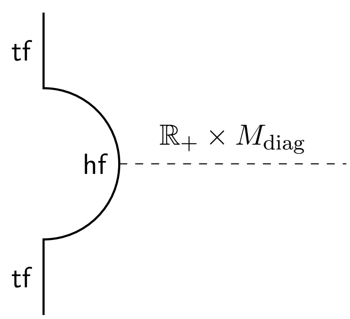

Figure 2.1: Heat space

Definition 2.4. The manifold (2.8) will be called the **heat space** of M, and we denote by

$$\beta_H : M^2_H \to \mathbb{R}_+ \times M^2 \tag{2.9}$$

the composite $\beta_H = \varrho \circ \beta$ of the blow-down map and the map (2.7).

The heat space has two boundary hypersurfaces; the front face of the blow-up, which we will refer to as the **heat face**, will be denoted $\mathsf{hf} \subset M^2_H$ and the other face (the lift of the original boundary face $\{0\} \times M^2$), which we refer to as the **temporal face** will be denoted by $\mathsf{tf} \subset M^2_H$. See Figure 2.1.

We fix once and for all **boundary defining functions** $\rho_{\mathsf{hf}}, \rho_{\mathsf{tf}} \in C^\infty\big(M^2_H; [0, \infty)\big)$, meaning smooth, non-negative functions such that $\rho_{\mathsf{hf}}^{-1}(0) = \mathsf{hf}$, $d\rho_{\mathsf{hf}}|_{\mathsf{hf}} \neq 0$, and similarly for ρ_{tf}.[4] As τ vanishes at both tf and hf, we may assume that

$$\tau = \rho_{\mathsf{tf}} \rho_{\mathsf{hf}}.$$

[4]Such a boundary defining function is unique up to multiplication by a strictly positive smooth function.

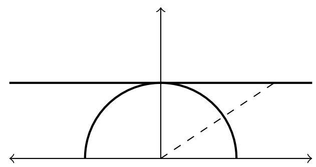

Figure 2.2: Radial compactification

Remark. In our consideration of the heat kernel on $\mathbb{R}^n$, the polar coordinate r is an example of a boundary defining function for hf, and ω_0 is a boundary defining function for tf.

Let us consider the structure of the heat face. As the inward pointing spherical normal bundle of $\{0\} \times M_{\text{diag}}$ it has the structure of a fiber bundle over $M = M_{\text{diag}}$.

Proposition 2.5. *The heat face* hf *is canonically diffeomorphic to the radial compactification of the tangent bundle over* M*:*

$$\mathsf{hf} \cong \overline{TM} \to M. \tag{2.10}$$

Proof. Recall that the radial compactification, $\overline{\mathbb{R}^n}$, of the vector space $\mathbb{R}^n$ is obtained by embedding $\mathbb{R}^n$ as $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^n \times \mathbb{R}_+$ and then identifying $\mathbb{R}^n$ with the open hemisphere by a variant of stereographic projection:

$$\mathbb{R}^n \ni v \longleftrightarrow \frac{1}{\sqrt{|v|^2+1}}(v,1) \in \mathring{\mathbb{S}}^n_+ \subset \mathbb{R}^n \times \mathbb{R}_+.$$

Then $\overline{\mathbb{R}^n}$ is identified with the closed hemisphere $\mathbb{S}^n_+$. (See Figure 2.2.)

If $E \to M$ is a vector bundle, then performing the same construction fiberwise in $E \oplus \mathbb{R}_+$ defines the fiberwise radial compactification $\overline{E} \to M$.

Now consider the (inward pointing) normal bundle of $\{0\} \times M_{\text{diag}} \subset \mathbb{R}_+ \times M^2$. Since we have a canonical temporal direction, this splits as

$$N_+(\{0\} \times M_{\text{diag}}) \cong \mathbb{R}_+ \oplus (N\, M_{\text{diag}}),$$

where $N\, M_{\text{diag}}$ denotes the normal bundle of $M_{\text{diag}} \subset M^2$, and this identifies the spherical normal bundle $S_+(\{0\} \times M_{\text{diag}})$ with the radial compactification $\overline{N\, M_{\text{diag}}}$. The rest of the claim then follows from the canonical isomorphism

$$NM_{\text{diag}} = T(M^2)/T(M_{\text{diag}}) = (TM \oplus TM)/TM \cong TM. \qquad \square$$

Rather than use polar coordinates, it is often more convenient to use **projective coordinates** on a blow-up. Suppose that $x = (x_1, \ldots, x_n)$ denotes local coordinates on a set $U \subset M$, and write (τ, x, y) for the corresponding coordinates on the intermediate space $(\mathbb{R}_+)_\tau \times U^2$ of (2.7). Then we have a local coordinate system

$$(\tau, x, \zeta) := \big(\tau, x, (x-y)/\tau\big), \quad \mathring{\mathsf{hf}} = \{\tau = 0\}\,, \tag{2.11}$$

on a neighborhood of the interior of $\mathsf{hf}|_U$, with respect to which the fibration (2.10) takes the form $(0, x, \zeta) \mapsto x$. Thus $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$ furnish Euclidean coordinates on the fibers of $\mathring{\mathsf{hf}} \cong TM \to M$.

2.3.3 Kernels

Motivated by the analysis of the heat kernel on $\mathbb{R}^n$ in §2.2, we will consider Schwartz kernels which are, up to an overall power of ρ_{hf}, smooth on M^2_H and rapidly vanishing with all derivatives at tf. For brevity, define

$$\Phi^k = \Phi^k(M; E, \ell) := \rho_{\mathsf{tf}}^\infty \rho_{\mathsf{hf}}^{-k} C^\infty\big(M^2_H; \mathrm{END}(E)\big) = \rho_{\mathsf{tf}}^\infty \tau^{-k} C^\infty\big(M^2_H; \mathrm{END}(E)\big), \quad k \in \mathbb{Z},$$

(the equality of spaces follows from the fact that $\tau = \rho_{\mathsf{tf}}\rho_{\mathsf{hf}}$), where $\mathrm{END}(E) = \mathrm{Hom}(\pi_2^* E, \pi_1^* E)$ and $\pi_i : M^2_H \to M$, $i = 1, 2$ denotes the composite of β_H and the corresponding projection to M. The notation $\rho^\infty C^\infty(M)$ is shorthand for the space $\bigcap_{n\in\mathbb{Z}} \rho^n C^\infty(M)$. We refer to k as the **order**.

Remarks.

- Based on the analysis of the heat kernel for $\partial_t + \Delta$ on $\mathbb{R}^n$, we expect that the heat kernel of $\partial_t + P$ will be a heat operator of order $n = \dim(M)$. Because of this, it would be natural to define the order to be $k - n - \ell$ rather than k, so that the "inverse" of $\partial_t + P$ has order $-\ell$. This is done in [Mel93], for instance. However we will persist with the convention above in the interest of simplicity.

- The heat operators are not pseudodifferential per se (though restricting to any fixed $t > 0$ they of course determine smoothing operators), but there are some similar features. The order defines a filtration of $\bigcup_{k\in\mathbb{Z}} \Phi^k$ with $k \leq l$ implying $\Phi^k \subseteq \Phi^l$. We will construct a parametrix order by order, solving away error terms of increasingly negative order.

Proposition 2.6. *Each $A \in \Phi^k$ defines an operator*

$$A : C^\infty(M; E) \to t^{(n-k)/\ell} C^\infty\big((\mathbb{R}_+)_{1/\ell} \times M; E\big) \subset t^{(n-k)/\ell} C^0\big(\mathbb{R}_+; C^\infty(M; E)\big)$$

where in the range space, $C^\infty\big((\mathbb{R}_+)_{1/\ell}\big)$ denotes a smooth function of $t^{1/\ell} = \tau$.

Proof. As an operator, A is given by pairing with the distributional kernel $\beta_*(A) \in C^{-\infty}\big((\mathbb{R}_+)_t \times M^2; \mathrm{END}(E)\big)$ via

$$Au = \int_M \beta_*(A)(x, y, t)\, u(y)\, \mathrm{dVol}_g(y) \tag{2.12}$$

For $t > 0$ this is evidently smooth, so it suffices to verify what happens near $t = 0$. Since A vanishes rapidly at tf, we need only consider a neighborhood of hf and may work locally.

Considering x and y now as local coordinates, we have $\mathrm{dVol}_g(y) = \omega(y)\,|dy|$, and then in terms of the coordinates (2.11) upstairs on M^2_H we have

$$\begin{aligned}\int \beta_*(A)(x,y,t)\,u(y)\,\omega(y)\,|dy| &= \int A(x,\zeta,\tau)\,u(x-\tau\zeta)\,\omega(x-\tau\zeta)\tau^n\,|d\zeta| \\ &= \int \tau^{-k}A_0(x,\zeta,\tau)\,u(x-\tau\zeta)\,\omega(x-\tau\zeta)\tau^n\,|d\zeta| \\ &= \tau^{n-k}\int A_0(x,\zeta,\tau)\,u(x-\tau\zeta)\,\omega(x-\tau\zeta)\,|d\zeta|\end{aligned} \tag{2.13}$$

where A_0 is smooth on M^2_H. By the smoothness of u and rapid decay of A_0 as $|\zeta| \to \infty$ (i.e., at tf), the integral converges to a smooth function of x and τ. The result then follows by setting $\tau = t^{1/\ell}$. $\square$

Setting $\tau = 0$ in the integral in formula (2.13) gives important information about the action of Φ^k at $t = 0$; namely the leading order (i.e., coefficient of $t^{(n-k)/\ell}$) of Au at $t = 0$ is given by

$$\big(t^{(k-n)/\ell}Au\big)|_{t=0} = c\,u, \quad c(x) = \int_{\mathbb{R}^n} A_0(x,\zeta,0)\,\omega(x)\,|d\zeta|\,, \tag{2.14}$$

where A_0 is the leading order coefficient of A with respect to τ at $\mathring{\mathsf{hf}} \cong TM$. This coefficient will play an important role below (it is the analogue in this context of the principal symbol for pseudodifferential operators), so in general we set

$$N(A) = N_k(A) := (\tau^k A)|_{\mathsf{hf}} \in \rho_{\mathsf{tf}}^\infty C^\infty\big(\mathsf{hf};\mathrm{End}(E)\big) \cong \mathcal{S}\big(TM;\mathrm{End}(E)\big), \quad A \in \Phi^k, \tag{2.15}$$

where we identify the smooth functions on $\mathsf{hf} \cong \overline{TM}$ vanishing rapidly at the boundary with the **Schwartz space** $\mathcal{S}(TM)$, consisting of smooth functions on TM whose restriction to each fiber satisfies the classical Schwartz estimates $\sup_\zeta \left|\zeta^\alpha \partial_\zeta^\beta u(\zeta)\right| < \infty$ for all $\alpha, \beta \in \mathbb{N}^n$.

Note that (2.14) makes invariant sense: we have an identification of $\mathring{\mathsf{hf}}$ with TM, and the Riemannian metric on M determines a Riemannian metric on TM, whose volume form at the fiber over x is precisely $\omega(x)\ |d\zeta|$, where (x,ζ) are standard coordinates on TM obtained from coordinates x on M. We denote by

$$\int_{\mathrm{fib}} : \mathcal{S}(TM) \to C^\infty(M)$$

the map given by integrating rapidly decaying functions over the fibers of TM with respect to this volume form. The result is of particular importance in the case $k = n$, so we record it as follows.

Corollary 2.7. *For $A \in \Phi^n$, let $N(A) \in \mathcal{S}\big(TM;\mathrm{END}(E)\big)$ denote the leading order coefficient of A at hf as above. Then the $t = 0$ restriction of A is the multiplication operator*

$$\begin{gathered} A|_{t=0} : C^\infty(M;E) \to C^\infty(M;E), \\ A|_{t=0}u = \Big(\int_{\mathrm{fib}} N(A)\Big)\,u \end{gathered}$$

In particular $A|_{t=0} = I$ if and only if $\int_{\mathrm{fib}} N(A) = I \in C^\infty\big(M;\mathrm{End}(E)\big)$.

We conclude that the heat kernel should be an element $H \in \Phi^n$ satisfying the initial condition that $N(H) = (\tau^n H)|_{\mathsf{hf}}$ has fiber integral $1 \in C^\infty\big(M; \mathrm{End}(E)\big)$.

2.3.4 Action of differential operators

Next we consider the action of $\partial_t + P$ on elements of Φ^k. Note that $\partial_t + P$ makes sense on $(\mathbb{R}_+)_t \times M^2$ (with P acting on the left factor of M^2), and we then want to pull it back to M^2_H via β. Now, vector fields (and by extension, differential operators) do not generally pull back under smooth maps, but since β is a diffeomorphism from the interior of M^2_H to the interior of $\mathbb{R}_+ \times M^2$, the lift is well-defined by continuous extension to the boundary. In fact to avoid singularities, it is convenient to lift $t(\partial_t + P)$ instead.

First consider $t\partial_t$. Lifting this with respect to (2.7), we have

$$t\partial_t \longmapsto \tau^\ell \frac{\partial \tau}{\partial t}\partial_\tau = \tau^\ell \Big(\frac{\partial t}{\partial \tau}\Big)^{-1}\partial_\tau = \tfrac{1}{\ell}\tau\partial_\tau.$$

Then in terms of the projective local coordinates (2.11) this further lifts to

$$\tfrac{1}{\ell}\tau\partial_\tau \longmapsto \tfrac{1}{\ell}\tau\Big(\partial_\tau + \textstyle\sum_j \dfrac{\partial \zeta_j}{\partial \tau}\partial_{\zeta_j}\Big) = \tfrac{1}{\ell}\tau\Big(\partial_\tau + \textstyle\sum_j -\dfrac{\zeta_j}{\tau^2}\partial_{\zeta_j}\Big) = \tfrac{1}{\ell}(\tau\partial_\tau - \zeta\cdot\partial_\zeta). \tag{2.16}$$

While we have employed local coordinates, the **radial vector field** $\zeta\cdot\partial_\zeta$ is in fact invariantly defined on the vector bundle $\mathring{\mathsf{hf}} \cong TM \to M$ as the infinitesimal generator of the scaling action by $(0,\infty)$.

Before lifting $tP \in \mathrm{Diff}^\ell(M;E)$, consider first the lift of a single vector field on M (i.e., a first order differential operator). Lifting $t^{1/\ell}V$, where V is a vector field given locally by $V = \sum_j a_j(x)\partial_{x_j}$, we have

$$t^{1/\ell}V \longmapsto \tau V = \tau\Big(\textstyle\sum_j a_j(x)\big(\partial_{x_j} + \dfrac{\partial \zeta_j}{\partial x_j}\partial_{\zeta_j}\big)\Big) = \textstyle\sum_j a_j(x)\partial_{\zeta_j} + \mathcal{O}(\tau).$$

Then if P is given locally by $P = \sum_{|\alpha|\le\ell} a_\alpha(x)\partial_x^\alpha$, it follows that tP lifts as

$$tP \longmapsto \sum_{|\alpha|=\ell} a_\alpha(x)\partial_\zeta^\alpha + \mathcal{O}(\tau). \tag{2.17}$$

In particular, modulo terms of order $\mathcal{O}(\tau)$, we only retain the principal part of P. Recall that one version of the principal symbol of a differential operator is as a homogeneous polynomial along the fibers of T^*M, varying smoothly with the base. Dually, $\sigma(P)$ may be considered as a *differential operator* on TM, which is differential only along the fibers and is translation invariant with respect to the linear structure. Indeed, taking a Fourier transform fiberwise identifies translation invariant differential operators on T_xM with polynomials on T_x^*M. The first term in (2.17) is well-defined invariantly as precisely this version of the principal symbol.

Proposition 2.8. *For $P \in \mathrm{Diff}^\ell(M;E)$, the lift of $t(\partial_t + P)$ to M_H^2 is a differential operator tangent to the boundary, whose restriction to $\mathring{\mathsf{hf}} \cong TM \to M$ has the form*

$$\sigma(P) - \tfrac{1}{\ell}\zeta \cdot \partial_\zeta \in \mathrm{Diff}^\ell_{\mathrm{fib}}(TM;\pi^* E),$$

where $\zeta \cdot \partial_\zeta$ is the radial vector field on TM, $\mathrm{Diff}^\ell_{\mathrm{fib}}(TM;E)$ denotes translation-invariant fiberwise differential operators on TM, and $\sigma(P) \in \mathrm{Diff}^\ell_{\mathrm{fib}}(TM;\pi^*E)$ denotes the principal symbol of P, regarded as a fiberwise differential operator on TM.

The action of $t(\partial_t + P)$ preserves Φ^k, and

$$N\big(t(\partial_t + P)A\big) = \big(\sigma(P) - \tfrac{1}{\ell}\zeta \cdot \partial_\zeta - \tfrac{k}{\ell}\big)N(A) \tag{2.18}$$

Proof. The first claim follows from the fact that $\tau = 0$ at hf, thus all terms of order $\mathcal{O}(\tau)$ in the local formulas above are vanishing there. To prove the second claim, we need to take into account the factor of τ in $A = \tau^{-k}N(A)$. To leading order the lift of tP commutes with τ^{-k}, so this factor doesn't contribute any additional terms. However, from (2.16), the lift of $t\partial_t$ is $\frac{1}{\ell}(\tau\partial_\tau - \zeta \cdot \partial_\zeta)$ and we have

$$\tau\partial_\tau(\tau^{-k}N(A)) = \tau^{-k}(\tau\partial_\tau - k)N(A),$$

which accounts for the additional factor in (2.18). $\square$

2.3.5 Heat Parametrix

The previous results give us a procedure to construct a parametrix $G \in \Phi^n$ such that $t(\partial_t + P)G = 0 \in \Phi^n/\Phi^{-\infty}$ in an iterative manner. We will do this in a way which emphasizes the similarities to the pseudodifferential parametrix construction for an elliptic operator.

Observe that $N(A) \in \rho_{\mathsf{tf}}^\infty C^\infty\big(\mathsf{hf}; \mathrm{End}(E)\big)$ characterizes $A \in \Phi^k$ modulo Φ^{k-1}. Said another way, the sequence

$$\Phi^{k-1} \hookrightarrow \Phi^k \xrightarrow{N} \mathcal{S}\big(TM; \mathrm{End}(E)\big) \tag{2.19}$$

is exact.

To construct a heat parametrix, we must initially find $G_0 \in \Phi^n$ such that

$$\begin{gathered}\big(\sigma(P) - \tfrac{1}{\ell}\zeta \cdot \partial_\zeta - \tfrac{n}{\ell}\big)\, N(G_0) = 0, \\ \int_{\mathrm{fib}} N(G_0) = I.\end{gathered} \tag{2.20}$$

Then from Proposition 2.8 and (2.19) it follows that $-R_0 := t(\partial_t + P)G_0 \in \Phi^{n-1}$.

By induction, suppose that we have found $G_j \in \Phi^{n-j}$, $j = 0, \ldots, k-1$ such that $t(\partial_t + P)(G_0 + \cdots + G_{k-1}) = -R_{k-1} \in \Phi^{n-k}$. We can then try to correct the error R_{k-1} by finding $G_k \in \Phi^{n-k}$ such that

$$\big(\sigma(P) - \tfrac{1}{\ell}\zeta \cdot \partial_\zeta - \tfrac{n-k}{\ell}\big)\, N(G_k) = N(R_{k-1}), \tag{2.21}$$

with no integral condition on the fibers.

Supposing that (2.20) and (2.21) can each be solved, we can construct $G \in \Phi^n$ such that $G \sim \sum_{j=0}^{\infty} G_j$, meaning that

$$G - \sum_{j=0}^{N} G_j \in \Phi^{n-N-1}$$

and then it follows that

$$\begin{aligned} t(\partial_t + P)G &= -R \in \Phi^{-\infty}, \\ G|_{t=0} &= I. \end{aligned}$$

Exercise 2.2. Prove the existence of the asymptotic sum $G \sim \sum_{j=0}^{\infty} G_j$. Up to some overall negative powers, this reduces to **Borel's lemma** in one dimension, which says that given any sequence $(a_j : j \in \mathbb{N}_0)$ of real numbers, there exists a smooth function $u(x) \in C_c^{\infty}(\mathbb{R})$ asymptotic to the power series

$$u(x) \sim \sum_{j=0}^{\infty} a_j x^j. \tag{2.22}$$

a compactly supported cutoff function $\phi \in C_c^{\infty}(\mathbb{R}; [0,1])$ with $\phi(x) \equiv 1$ for $|x| \leq 1$ and $\phi(x) \equiv 0$ for $|x| \geq 2$. Then show that, for a sequence $\varepsilon_j \searrow 0$, the series

$$u(x) = \sum_{j=0}^{\infty} a_j x^j \phi(x/\varepsilon_j)$$

converges pointwise (since for all $x > 0$ it is finite) and has the required asymptotic property.

To actually solve (2.20) and (2.21) requires some detailed consideration of the principal symbol of P, hence we will now specialize to the case of primary interest.

2.3.6 Parametrix for Laplace-type operators

Definition 2.9. Say an operator $P \in \mathrm{Diff}^2(M; E)$ is a **Laplace-type operator** if its principal symbol is

$$\sigma(P)(x, \xi) = |\xi|^2 I \in C^{\infty}\big(T^*M; \mathrm{End}(E)\big),$$

where $|\xi|^2 = g(\xi, \xi)$ is computed with respect to the Riemannian metric on M. (In the representation of $\sigma(P)$ on the cosphere bundle, the condition is equivalent to $\sigma(P)(x, \xi) = I$, and in the representation of $\sigma(P)$ as a fiberwise translation-invariant differential operator, the condition is equivalent to $\sigma(P) = \Delta_\zeta$, the fiberwise Laplacian with respect to the induced Riemannian metric on TM.) We do not require P to be self-adjoint.

For a Laplace-type operator, we are reduced to considering solvability of the differential operators

$$\Delta_\zeta - \tfrac{1}{2}(\zeta \cdot \partial_\zeta + n - k) \in \mathrm{Diff}^2(\mathbb{R}^n), \quad k \in \mathbb{N}_0.$$

Observe that the Fourier transform intertwines this operator with the operator

$$|\xi|^2 + \tfrac{1}{2}\xi \cdot \partial_\xi + \tfrac{k}{2} \in \mathrm{Diff}^1(\mathbb{R}^n)$$

solutions of which can be explicitly determined.

Indeed, let $g_j(\zeta) \in \mathcal{S}(\mathbb{R}^n)$ denote the restriction of $N(G_j)$ to a given fiber of TM (really this is valued in matrices, but since we always work with scalar multiples of the identity matrix, we shall omit this from the notation), and $\widehat{g}_j(\xi)$ its Fourier transform. Then (2.20) is equivalent to

$$(\xi \cdot \partial_\xi + 2\,|\xi|^2)\widehat{g}_0(\xi) = 0, \quad \widehat{g}_0 \in \mathcal{S}(\mathbb{R}^n), \quad \widehat{g}_0(0) = 1.$$

(The last condition is equivalent to the integral condition $\int_{\mathbb{R}^n} g_0(\zeta)\,d\zeta = 1$.) This has the explicit solution

$$\widehat{g}_0(\xi) = e^{-|\xi|^2} \implies g_0(\zeta) = (4\pi)^{-n/2} e^{-|\zeta|^2/4},$$

as can be directly verified, and can be derived by solving the ODE $\partial_t u(t) = -2tu(t)$, $u(0) = 1$, along each radial line. Clearly $g_0 \in \mathcal{S}(\mathbb{R}^n)$, and by uniqueness of solutions to first order ODE, is the unique solution.

The higher order problem (2.21), which is equivalent to

$$(\xi \cdot \partial_\xi + 2\,|\xi|^2 + k)\widehat{g}_k(\xi) = 2\widehat{r}_{k-1}(\xi) \in \mathcal{S}(\mathbb{R}^n), \tag{2.23}$$

can also be solved explicitly. Indeed, along a radial line, the homogeneous ODE $\partial_t u = (-2t + k/t)u$ has the fundamental solution $t^k e^{-t^2} u(0)$, convolution with which solves the inhomogeneous problem. Thus (2.23) has the unique solution

$$\begin{aligned} \widehat{g}_k(t\xi) &= 2\int_0^t (t-s)^k e^{-(t-s)^2} \widehat{r}_{k-1}(s\xi)\,ds, \quad \text{or} \\ \widehat{g}_k(\xi) &= 2\int_0^1 |\xi|^{k+1}\,(1-r)^k e^{-(1-r)^2|\xi|^2} \widehat{r}_{k-1}(r\xi)\,dr \in \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

Combining these results with the induction in §2.3.5, we conclude

Proposition 2.10. *Let $P \in \mathrm{Diff}^2(M;E)$ be a Laplace-type operator. Then there exists a parametrix $G \in \Phi^n = \rho_{\mathsf{tf}}^\infty \tau^{-n} C^\infty\big(M_H^2; \mathrm{END}(E)\big)$ such that*

$$\begin{gathered} t(\partial_t + P)G = -R \in \Phi^{-\infty}, \\ G|_{t=0} = I. \end{gathered} \tag{2.24}$$

In particular, $N(G)(\zeta, x) = (4\pi)^{-n/2} e^{-|\zeta|^2/4} I \in \mathcal{S}\big(TM; \mathrm{End}(E)\big)$.

2.3.7 True solution

It remains to correct the final error R in (2.24). For this we consider the action of Schwartz kernels in $C^{-\infty}(\mathbb{R}_+ \times M^2)$ as **convolution operators**:[5,6]

$$C^{-\infty}(\mathbb{R}_+ \times M^2) \ni A* : C^\infty(\mathbb{R}_+ \times M) \to C^\infty(\mathbb{R}_+ \times M),$$
$$(A * u)(t, x) = \int_0^t \int_M A(t - s, x, y)u(s, y) \,\mathrm{dVol}_g(y)\, ds =: \int_0^t [Au(s)](t - s)\, ds.$$

The composition $C* = (A * B)*$ of convolution operators A and B is given by the Schwartz kernel

$$C(t, x, z) = \int_0^t \int_M A(t - s, x, y)B(s, y, z) \,\mathrm{dVol}_g(y)\, ds = \int_0^t A(t - s)B(s)\, ds.$$

The convolution point of view is natural in the context of initial value problems; observe that for any A whose limit as an operator on M is appropriately well-defined at $t = 0$, we have

$$\begin{aligned} \partial_t(A * u) &= \partial_t \int_0^t [Au(s)](t - s)\, ds \\ &= [Au(t)](0) + \int_0^t [\partial_t Au(s)](t - s)\, ds \\ &= A|_{t=0}u|_{t=t} + (\partial_t A) * u. \end{aligned} \tag{2.25}$$

(The first term comes from the fundamental theorem of calculus, with ∂_t differentiating the upper limit t, and in the second term ∂_t has been taken inside the integral.) If you prefer, in more spelled out notation,

$$\begin{aligned} \partial_t(A * u) &= \partial_t \int_0^t \int_M A(t - s, x, y)u(s, y) \,\mathrm{dVol}_g(y)\, ds \\ &= \int_M A(0, x, y)u(t, y) \,\mathrm{dVol}_g(y) + \int_0^t \partial_t A(t - s, x, y)u(s, y) \,\mathrm{dVol}_g(y)\, ds. \end{aligned}$$

In particular, the heat kernel conditions $(\partial_t + P)H = 0$, $H|_{t=0} = I$ are equivalent to the condition that

$$(\partial_t + P)H* = H|_{t=0} + \big((\partial_t + P)H\big)* = I$$

as a convolution operator. (Perhaps we should write $I*$ rather than I to emphasize the difference between the identity convolution operator $I* = \delta_0(t)\delta_{\mathrm{diag}}(x, y)$ and the time-independent identity operator $I = \delta_{\mathrm{diag}}(x, y)$, but we shall not do so.)

[5] Hopefully there is no confusion arising from our identification of operators with their Schwartz kernels! As a general rule if we do not specify the spatial coordinates x or y, then adjacency should be read as operator composition.

[6] In general we should make some restrictions on either the distributions or the smooth functions on $\mathbb{R}_+$ at $t = 0$ so that the action below is well-defined, but we will ignore this point since we do not require the most general statement.

Proposition 2.11. *As a convolution operator, the heat parametrix $G \in \Phi^n$ of Proposition 2.10 satisfies*

$$(\partial_t + P)\, G* = I - R'* \tag{2.26}$$

where $R' = t^{-1}R \in \Phi^{-\infty}$.

Proof. This follows from the condition $G|_{t=0} = I$ as computed above, along with the fact that $t(\partial_t + P)G = -R$, so $(\partial_t + P)G = -t^{-1}R =: -R'$. Pulling t^{-1} back to M^2_H gives $R' = \rho_{\mathsf{tf}}^{-2}\rho_{\mathsf{hf}}^{-2}R$, but since $R \in \rho_{\mathsf{tf}}^{\infty}\rho_{\mathsf{hf}}^{\infty}C^\infty\big(M^2_H; \mathrm{END}(E)\big)$ these factors may be absorbed. □

Now, the residual space $\Phi^{-\infty}$ is easy to characterize:

$$\Phi^{-\infty} = \rho_{\mathsf{tf}}^{\infty}\rho_{\mathsf{hf}}^{\infty}C^\infty\big(M^2_H; \mathrm{END}(E)\big)$$

consists of smooth kernels which vanish rapidly at all boundary faces, and in fact this equivalent to

$$\Phi^{-\infty} \equiv \tau^\infty C^\infty\big((\mathbb{R}_+)_\tau \times M^2; \mathrm{END}(E)\big) = t^\infty C^\infty\big((\mathbb{R}_+)_t \times M^2; \mathrm{END}(E)\big),$$

the space of kernels on the original space $\mathbb{R}_+ \times M^2$ which are rapidly vanishing in t.

Proposition 2.12. *If $A \in \Phi^{-\infty} = t^\infty C^\infty\big(\mathbb{R}_+ \times M^2; \mathrm{END}(E)\big)$, then the convolution operator $I - A*$ is invertible, with inverse*

$$I - S* = I + \sum_{j=1}^{\infty}(A*)^j, \quad S* \in \Phi^{-\infty} \tag{2.27}$$

Remark. The infinite series in (2.27) is called a **Volterra series** and $A*$ for operators $A \in \Phi^{-\infty}$ are sometimes referred to as **Volterra operators**. The series can be written explicitly in terms of integrals over simplices:

$$\sum_{j=0}^{\infty}(A*)^j = \sum_{j=0}^{\infty}\int_{\triangle^j} A(t_j)\cdots A(t_1)\,dt_1\cdots dt_j, \quad \triangle^j = \{(t_1,\dots,t_j) : 0 \le t_1 \le \cdots \le t_j \le t\}$$

Proof. Fix $T > 0$. Direct estimation shows that, if B_1 and B_2 are general convolution operators such that

$$|B_1(t,x,y)| \le C_k \frac{t^k}{k!}, \quad |B_2(t,x,y)| \le C_0, \quad t \in [0,T],$$

then the kernel $B = B_1 * B_2$ satisfies the pointwise estimate

$$|B(t,x,y)| \le C_0 C_k \operatorname{Vol}(M)\frac{t^{k+1}}{(k+1)!}, \quad t \in [0,T].$$

By assumption A satisfies estimates of the form $|A(t,x,y)| \le C_k t^k/k!$ for any k, hence by induction it follows that the kernel, A_j, of $(A*)^j$ satisfies

$$|A_j(t,x,y)| \le C_k\big(C_0 \operatorname{Vol}(M)\big)^{j-1}\frac{t^{k+j-1}}{(k+j-1)!}, \quad t \in [0,T].$$

By the ratio test it follows that the Volterra series $\sum_j (A*)^j$ converges pointwise for all $t \leq T$, and since T was arbitrary, for all t. For any $m \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^n$, the kernel $\partial_t^m \partial_x^\alpha \partial_y^\beta A$ satisfies similar estimates, so it follows that $\sum_j (A*)^j$ is in $C^\infty\big(\mathbb{R}_+ \times M^2; \mathrm{END}(E)\big)$. Furthermore, since k was arbitrary, it follows that $S = -\sum_{j=1}^\infty (A*)^j$ is rapidly vanishing in t, which completes the claim. $\square$

In light of Proposition 2.12, we let $S \in \Phi^{-\infty}$ be determined by $I - S* = (I - R'*)^{-1}$ and then we may define the **true heat kernel** via

$$H := G * (I - R'*)^{-1} = G * (I - S*) = G - G * S. \tag{2.28}$$

Then H satisfies

$$(\partial_t + P)H* = I \tag{2.29}$$

(equivalently, $(\partial_t + P)H = 0$ and $H|_{t=0} = I$).

Theorem 2.13 (Heat kernel for a Laplace-type operator)**.** *For a Laplace-type operator $P \in \mathrm{Diff}^2(M; E)$, the kernel* (2.28) *constructed above is the unique solution to* (2.29)*; in particular, if P is a positive self-adjoint operator, then $H = e^{-tP}$ agrees with the heat kernel as defined via the spectral measure. Moreover,*

$$H \in \Phi^n = \rho_{\mathsf{tf}}^\infty \tau^{-n} C^\infty\big(M_H^2; \mathrm{END}(E)\big),$$

and $H - G \in \Phi^{-\infty}$, so the asymptotic expansions of H and G at $\mathsf{hf} \subset M_H^2$ agree. In particular

$$N(H)(\zeta, x) = (\tau^n H)|_{\mathring{\mathsf{hf}} \cong TM} = (4\pi)^{-n/2} e^{-|\zeta|^2/4} \in \mathcal{S}\big(TM; \mathrm{End}(E)\big).$$

Proof. The uniqueness of H is equivalent to triviality of solutions with vanishing initial data:

$$(\partial_t + P)u = 0, \quad u(0) = 0 \implies u \equiv 0. \tag{2.30}$$

To prove (2.30), suppose u solves $(\partial_t + P)u = 0$ with $u(0) = 0$. First note that $\partial_t u|_{t=0} = -Pu|_{t=0} = 0$, and then using $\partial_t(\partial_t + P) = (\partial_t + P)\partial_t$ and induction it follows that all derivatives of u vanish at $t = 0$. Thus u can be extended to a smooth section $u \in C^\infty(\mathbb{R} \times M; E)$ with $u = 0$ for $t \leq 0$. Next, observe that the $L^2(\mathbb{R} \times M; E)$ adjoint of $\partial_t + P$ is $-\partial_t + P^*$, and that P^* is a Laplace-type operator since P is. The operator $-\partial_t + P^*$ is the time reversed heat flow of P^*, so from the construction above of a heat kernel above it follows that for any $\phi \in C_c^\infty(\mathbb{R} \times M; E)$, say with $\phi = 0$ for $t > T$, we can solve $(-\partial_t + P^*)v = \phi$ with $v = 0$ for $t > T$. Then

$$(u, \phi)_{L^2(\mathbb{R} \times M; E)} = \big(u, (-\partial_t + P^*)v\big) = \big((\partial_t + P)u, v\big) = 0$$

and since ϕ was arbitrary, we conclude $u = 0$.

It remains to prove that $G - H = G * S$ is in the residual space $\Phi^{-\infty}$. First, from the mapping property in Proposition 2.6, for $S \in \Phi^{-\infty}$ (or more generally any kernel which is smooth in t) it follows that $(s,t) \mapsto G(s)S(s-t) \in C^\infty\big(M^2; \mathrm{END}(E)\big)$ is continuous uniformly in both s and t. Then

$$G * S \in tC^0\big(\mathbb{R}_+; C^\infty(M^2; \mathrm{END}(E))\big),$$

since $G * S$ is the integral from 0 to t in s of a uniformly continuous function of s and t (with values in $C^\infty\big(M^2; \mathrm{END}(E)\big)$. Then consider $\partial_t(G * S)$. By (2.25) and the fact that convolution is symmetric in t (i.e., $\int_0^t G(t-s)S(s)\,ds = \int_0^t G(s)S(t-s)\,ds$), this is equal to

$$\partial_t(G * S) = S + G * (\partial_t S),$$

which is again in $tC^0\big(\mathbb{R}_+; C^\infty(M^2; \mathrm{END}(E))\big)$ since $\partial_t S$ is of the same type. By iteration, it follows that $G * S$ is smooth and rapidly decreasing, i.e., in $\Phi^{-\infty}$. □

2.4 Trace class operators

Let us briefly digress to review the subject of trace-class operators. The goal is to characterize those operators A on an infinite dimensional, separable, Hilbert space $\mathcal{H}$ such that the quantity

$$\mathrm{Tr}(A) = \sum_i (Ae_i, e_i)$$

is well-defined and independent of the choice of orthonormal basis $\{e_i\}$. All the ways of doing this are a bit fiddly, and the path we shall take is via so-called *Hilbert-Schmidt* operators.

Definition 2.14. A bounded operator A on $\mathcal{H}$ is **Hilbert-Schmidt** if, for some choice of orthonormal basis $\{e_i\}$, the quantity

$$\|A\|_{HS}^2 := \sum_i \|Ae_i\|^2 < \infty \tag{2.31}$$

is finite. Equivalently, if we denote by $a_{ij} = (Ae_j, e_i)$ the *matrix coefficients* of A, then (2.31) is equivalent to the sum

$$\|A\|_{HS}^2 = \sum_{i,j} |a_{ij}|^2 < \infty. \tag{2.32}$$

We denote the set of Hilbert-Schmidt operators on $\mathcal{H}$ by $\mathcal{B}_2(\mathcal{H})$.

Proposition 2.15. *The Hilbert-Schmidt operators form a 2-sided ideal in the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$, and* (2.31) *is independent of the choice of basis. Hilbert-Schmidt operators are compact operators.*

Proof. From (2.32) it is clear that A is Hilbert-Schmidt if and only if A^* is, and $\|A\|_{HS} = \|A^*\|_{HS}$. From (2.31) it is clear that if A is Hilbert-Schmidt and B is bounded, then BA is Hilbert-Schmidt, with

$$\|BA\|_{HS} \le \|B\|\,\|A\|_{HS}\,,$$

and then so is $AB = (B^*A^*)^*$ by the first observation. It follows that the Hilbert-Schmidt operators form a 2-sided ideal. Note that if $U \in \mathcal{B}(\mathcal{H})$ is unitary, then

$$\|UA\|_{HS} = \|AU^*\|_{HS} = \|A\|_{HS}\,,$$

from which it follows that $\|A\|_{HS}$ is independent of the basis.

From (2.31) it is easy to show that $A \in \mathcal{B}_2(\mathcal{H})$ is the norm limit of the finite rank operators $A_n = A\Pi_n$, where $\Pi_n = \sum_{i=1}^n (e_i, \cdot)\, e_i$ is the projection onto the space spanned by the first n vectors in the basis. It follows that A is compact. □

In particular, if A is a self-adjoint Hilbert-Schmidt operator, then $\|A\|_{HS}$ may be computed with respect to the orthonormal basis of eigenvectors afforded by Theorem 1.47, from which it follows that

$$A \in \mathcal{B}_2(\mathcal{H}),\ A = A^* \implies \|A\|_{HS} = \Big(\sum_j \lambda_j^2\Big)^{1/2},$$

where $\{\lambda_j\}$ is the sequence of eigenvalues of A with multiplicity.

In fact, the H-S norm is associated to the **Hilbert-Schmidt inner product**

$$(A,B)_{HS} = \sum_i (Ae_i, Be_i) = \sum_i (B^*Ae_i, e_i),$$

with respect to which $\mathcal{B}_2(\mathcal{H})$ has the structure of a Hilbert space. The inner product is related to the norm via the the usual *polarization identity*

$$(A,B)_{HS} = \frac{1}{4}\sum_{k=0}^{4} i^k \big\|A + i^k B\big\|_{HS}^2. \tag{2.33}$$

Replacing A and B by their adjoints and using $A^* + i^k B^* = (A + (-i)^k B^*)$ in (2.33) leads to the identity

$$(A,B)_{HS} = (A^*, B^*)_{HS}. \tag{2.34}$$

The H-S inner product leads to the initial definition of trace-class operators.

Definition 2.16. An operator $C \in \mathcal{B}(\mathcal{H})$ is **trace-class** if it has the form $C = B^*A$ for Hilbert-Schmidt operators $A, B \in \mathcal{B}_2(\mathcal{H})$. The **trace** of $C = B^*A$ is defined by

$$\mathrm{Tr}(C) = \sum_i (Ce_i, e_i) = (A,B)_{HS}, \tag{2.35}$$

which is therefore independent of the choice of basis $\{e_i\}$ (and of the choice of A and B). The set of trace-class operators is denoted $\mathcal{B}_1(\mathcal{H})$.

It is not quite obvious that $\mathcal{B}_1(\mathcal{H})$ is a linear subspace, but this follows by identifying $\mathcal{H} \cong \mathcal{H} \oplus \mathcal{H}$ and then if $C_i = B_i^* A_i$ for $i = 1, 2$, then

$$C_1 + C_2 = \begin{pmatrix} B_1^* & B_2^* \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

Then from the ideal property of $\mathcal{B}_2(\mathcal{H})$, it follows that $\mathcal{B}_1(\mathcal{H})$ is again a 2-sided ideal in $\mathcal{B}(\mathcal{H})$, containing $\mathcal{B}_2(\mathcal{H})$ and consisting of compact operators. Thus if we denote by $\mathcal{B}_0(\mathcal{H})$ the compact operators, then we have inclusions of ideals

$$\mathcal{B}_2(\mathcal{H}) \subset \mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}_0(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}).$$

In particular, since $C \in \mathcal{B}_1(\mathcal{H})$ is compact, by the spectral theorem, the operator $|C| := (C^*C)^{1/2}$ is well-defined, with spectral resolution

$$|C| = \sum_j \mu_j (e_i, \cdot) e_i,$$

where $\{\mu_i^2\} \subset \mathbb{R}_+$ are the eigenvalues of the positive compact operator C^*C. We refer to the μ_j as the **singular values** of C. This leads to an alternative characterization of trace-class operators:

Proposition 2.17. *An operator C is trace-class if and only if C is compact and*

$$\|C\|_1 := \sum_j \mu_j < \infty. \tag{2.36}$$

Proof. Suppose that C is compact and (2.36) holds. From

$$\||C|\, e_i\|^2 = \big((C^*C)^{1/2}e_i, (C^*C)^{1/2}e_i\big) = \big((C^*C)e_i, e_i\big) = \big(Ce_i, Ce_i\big) = \|Ce_i\|^2$$

it follows that there is an isometry between the range of C and the range of $|C|$, thus

$$C = U\,|C|\,, \quad |C| = WC \tag{2.37}$$

for some partial isometries U and W. Let $D = |C|^{1/2} = (C^*C)^{1/4}$, defined again by the spectral theorem, which has eigenvalues $\{\sqrt{\mu_j}\}$. The hypothesis (2.36) implies that D is Hilbert-Schmidt, and then from (2.37),

$$C = U\,|C| = UD^2$$

is the product of Hilbert-Schmidt operators UD and $D^* = D$.

Conversely, if $C = B^*A$ is trace class, then it follows from (2.37) that $|C| = (WB^*)A = (BW^*)^*A$ is trace class, and computing the trace using the basis $\{e_i\}$ of eigenvectors for $|C|$ leads to $\|C\|_1 = \mathrm{Tr}(|C|)$, proving (2.36). □

Remark. In fact $\|C\|_1 = \mathrm{Tr}(|C|)$ gives a norm on $\mathcal{B}_1(\mathcal{H})$ with respect to which it enjoys the structure of a Banach space. More generally, for any $p \in [1, \infty)$, the *Schatten class*

$$\mathcal{B}_p(\mathcal{H}) \ni A \iff A \text{ compact, and } \|A\|_p = \Big(\sum_j \mu_j^p\Big)^{1/p} < \infty$$

defines an ideal within the compact operators, such that $\mathcal{B}_p(\mathcal{H}) \subseteq \mathcal{B}_q(\mathcal{H})$ if $q \leq p$, and these are all complete with respect to the norms $\|A\|_p = \mathrm{Tr}(|A|^p)^{1/p}$. Furthermore, there is a Hölder-type estimate $\|AB\|_1 \leq \|A\|_p \|B\|_q$ when $\frac{1}{p} + \frac{1}{q} = 1$.

Note that, in the case that $C \in \mathcal{B}_1(\mathcal{H})$ is diagonalizable (say if C is self-adjoint, or more generally normal), with eigenvectors $\{\lambda_j\}$, then $\mu_j = |\lambda_j|$ and the trace of C is given by

$$\mathrm{Tr}(C) = \sum_j \lambda_j. \tag{2.38}$$

Proposition 2.18. *If C is trace-class and $D \in \mathcal{B}(\mathcal{H})$ is an arbitrary bounded operator, then* $\mathrm{Tr}([C, D]) = 0$.

Proof. The commutator $[C, D] = CD - DC$ is trace-class since $\mathcal{B}_1(\mathcal{H})$ is an ideal. Write $C = B^*A$ for $A, B \in \mathcal{B}_2(\mathcal{H})$. Then,

$$\begin{aligned}\mathrm{Tr}(CD) = (AD, B)_{HS} &= \overline{(B, AD)}_{HS} = \overline{(B^*, D^*A^*)}_{HS} = \overline{\mathrm{Tr}(ADB^*)} \\ &= \overline{(DB^*, A^*)}_{HS} = \overline{(BD^*, A)}_{HS} = (A, BD^*)_{HS} = \mathrm{Tr}(DB^*A) = \mathrm{Tr}(DC).\end{aligned}$$

□

Remark. The commutator identity $\mathrm{Tr}([C, D]) = 0$ often holds even when D is an *unbounded* operator; Indeed, the proof above holds provided that AD and DB^* are well-defined and Hilbert-Schmidt. This is often true in the cases of interest, for instance when D is a positive order (pseudo)differential operator and A and B can be taken to be smoothing operators (see below).

2.4.1 Integral kernels

We consider now the case that $\mathcal{H} = L^2(M)$ (or more generally $L^2(M; E)$, though we shall omit the bundles for notational convenience), for a Riemannian manifold (M, g). Fix an orthonormal basis $\{e_i(x)\}$. Then it is easy to show (using Fubini's theorem) that

$$\left\{e_i(x)\, e_j^*(y) : (i, j) \in \mathbb{N}^2\right\} \text{ is an orthonormal basis for } L^2(M \times M).$$

From this we get a nice characterization of the Hilbert-Schmidt operators on $L^2(M)$ via their Schwartz kernels.

Proposition 2.19. $A \in \mathcal{B}_2\big(L^2(M)\big)$ *if and only if A has Schwartz kernel*

$$K_A \in L^2(M \times M),$$

and the map $\mathcal{B}_2\big(L^2(M)\big) \to L^2(M \times M)$, $A \mapsto K_A$ is an isometry: $\|A\|_{HS} = \|K_A\|_{L^2(M\times M)}$.

Proof. If $K_A \in L^2(M \times M)$, then it defines a bounded operator, A, on $L^2(M)$, and the Hilbert-Schmidt norm of A is

$$\|A\|_{HS}^2 = \sum_i \|Ae_i\|_{L^2(M)}^2 = \sum_{i,j} (Ae_i, e_j)^2$$

$$= \sum_{i,j} \Big| \int_{M\times M} K_A(x, y)\, e_i(y)\, e_j^*(x)\, \mathrm{dVol}_x\, \mathrm{dVol}_y \Big|^2 = \sum_{i,j} (K_A, e_i \otimes e_j^*)_{L^2(M\times M)}^2 = \|K_A\|_{L^2(M\times M)}^2.$$

Conversely, if $A \in \mathcal{B}_2\big(L^2(M)\big)$ then it has a kernel representation $K_A(x,y) = \sum_{i,j} a_{ij}\, e_i(x)\, e_j^*(y)$ and

$$\begin{aligned} \|K_A\|^2_{L^2(M\times M)} &= \int_{M\times M} \sum_{i,j} \big|a_{ij}\, e_i(x)\, e_j^*(y)\big|^2 \, \mathrm{dVol}_x \, \mathrm{dVol}_y \\ &= \sum_{i,j} \int_{M\times M} \big|a_{ij}\, e_i(x)\, e_j^*(y)\big|^2 \, \mathrm{dVol}_x \, \mathrm{dVol}_y = \sum_{i,j} |a_{ij}|^2 = \|A\|^2_{HS}\,. \quad \square \end{aligned}$$

This leads to the following result, which the author has seen referred to as *Lidskii's Theorem*, though other sources use that name to refer to the assertion (2.38).

Theorem 2.20 (Lidksii's theorem)**.** *If A is a trace class operator on $L^2(M)$, and the restriction of the Schwartz kernel of A to the diagonal $M_{\mathrm{diag}} \subset M \times M$ is well-defined, then*

$$\mathrm{Tr}\, A = \int_M A(x,x)\, \mathrm{dVol}_x$$

Remark. Observe that both hypotheses are necessary. There exist non-trace class operators whose Schwartz kernels admit a well-defined restriction to the diagonal with $\int_M A(x,x)\, dx < \infty$, and there exist trace-class operators whose restriction to the diagonal is ill-defined. The theorem is most useful in the case that the kernel of A lies in $C^0(M \times M)$, and then both hypotheses are satisfied since $A \in L^2(M \times M) = \mathcal{B}_2\big(L^2(M)\big)$.

Proof. Let $A = B^*C$ for Hilbert-Schmidt operators B and C. In terms of kernels,

$$A(x,y) = \int_M B^*(x,z)\, C(z,y)\, \mathrm{dVol}_z. = \int_M \overline{B}(z,x)\, C(z,y)\, \mathrm{dVol}_z. \tag{2.39}$$

By Proposition 2.19, $\mathrm{Tr}\, A = (C,B)_{HS} = (C,B)_{L^2(M\times M)}$, so

$$\mathrm{Tr}\, A = \int_{M\times M} \overline{B}(z,x)\, C(z,x)\, \mathrm{dVol}_z\, \mathrm{dVol}_x = \int_M A(x,x)\, \mathrm{dVol}_x$$

by (2.39) and the hypothesis that $A|_M$ is well-defined. $\square$

Corollary 2.21. *Let $A \in \Psi^{-\infty}(M;E)$ be a smoothing operator. Then A is a trace class operator on $L^2(M;E)$ with*

$$\mathrm{Tr}\, A = \int_M \mathrm{tr}\, A(x,x)\, dx,$$

where $\mathrm{tr} : C^\infty(M; \mathrm{End}(E)) \to C^\infty(M)$ denotes the fiberwise trace.

2.5 Heat Trace and Weyl asymptotics

Returning to the heat equation for a Laplace-type operator $P \in \mathrm{Diff}^2(M;E)$, we may apply Lidskii's theorem to the heat kernel $H = e^{-tP}$ for any fixed $t > 0$, since this has kernel in $C^\infty\big(M^2; \mathrm{END}(E)\big)$. We conclude that

$$\mathrm{Tr}\, e^{-tP} = \int_{M_{\mathrm{diag}}} \mathrm{tr}\, H(t,x,x)\, \mathrm{dVol}_g(x), \quad t > 0,$$

where the integral is taken over $\{t\} \times M_{\mathrm{diag}} \subset M_H^2$. Since H (and therefore also $\mathrm{tr}\, H$) has a complete asymptotic expansion on M_H^2, it follows that $\mathrm{Tr}\, e^{-tP}$ has an asymptotic expansion, which may be computed by integrating over the "lifted diagonal" $\mathbb{R}_+ \times M_{\mathrm{diag}} \subset M_H^2$. The latter intersects the heat face $\mathring{\mathsf{hf}} \cong TM$ at the zero section, and we conclude:

Proposition 2.22. *The heat trace* $\mathrm{Tr}\, e^{-tP}$ *has a complete* **short-time asymptotic expansion** *as* $t \searrow 0$ *of the form*

$$\mathrm{Tr}\, e^{-tP} \sim t^{-n/2} \sum_{k=0}^{\infty} a_k t^{k/2}, \tag{2.40}$$

with $a_0 = \int_M \mathrm{tr}\, N(H)(0,x)\, \mathrm{dVol}_g = (4\pi)^{-n/2}\, \mathrm{Vol}(M)\, \mathrm{Rank}(E)$.

Remarks.

- By taking parity considerations into account with respect to the involution $\zeta \mapsto -\zeta$ on hf, it is possible to show that all of the terms of order $\mathcal{O}(\tau^{2n+1})$ in the asymptotic expansion of H are odd with respect to the involution, hence evaluate to 0 at $\zeta = 0$. In particular, all the coefficients a_k in (2.40) with k odd are vanishing.

- By working a bit harder, it is possible in principle to compute the lower order asymptotics as well, though this quickly becomes quite difficult. For instance, for $P = \Delta$, the scalar Laplacian,
$$a_2 = (4\pi)^{-n/2} \frac{1}{6} \int_M \mathrm{scal}\, \mathrm{dVol}_g$$
is proportional to the integral of the scalar curvature of g, and a_4 involves the square integrals of the scalar, Ricci and full curvature tensors. In general the coefficients are integrals of polynomials in covariant derivatives of the curvature tensors of g, but they quickly get out of hand: see [Gil08] for a general survey (the formula for a_6 on page 5 occupies nine lines). The exact formulas are typically computed by writing the a_{2n} in terms of polynomials with undetermined coefficients, and then computing the explicit heat kernel in sufficiently many special cases to fix the coefficients.

While we are on the topic, let us digress for a moment to consider the asymptotic expansion of the heat trace as $t \to +\infty$. Knowing that the heat kernel is trace class for each fixed $t > 0$, we can read this asymptotic expansion off from the spectral resolution $e^{-tP} = \sum_{\lambda_j \in \mathrm{Spec}(P)} e^{-t\lambda} e_j(x) e_j^*(y)$:

$$\mathrm{Tr}\, e^{-tP} = \sum_{\lambda_j \in \mathrm{Spec}(P)} e^{-t\lambda_j} N_j, \quad N_j = \dim \mathrm{Null}(P - \lambda_j).$$

The following is then immediate:

Proposition 2.23. *The* **long-time asymptotic expansion** *of the heat trace is*

$$\mathrm{Tr}\, e^{-tP} = \dim \mathrm{Null}(P) + \mathcal{O}(e^{-t\lambda_1}), \quad t \to +\infty. \tag{2.41}$$

Returning to the discussion started near the beginning of this chapter, now that we know that e^{-tP} is trace-class for $t > 0$, it follows that the computation

$$\mathrm{Tr}\, e^{-tP} = \mathrm{Tr} \int_{\mathbb{R}_+} e^{-t\lambda}\, dE(\lambda) = \int_{\mathbb{R}_+} e^{-t\lambda}\, d\mu(\lambda)$$

is justified, where we recall that $dE(\lambda)$ is the spectral measure of P and $d\mu(\lambda) = \mathrm{Tr}\, dE(\lambda)$ is the counting measure on eigenvalues of P. We can now make use of the following result to get information about the asymptotics of $d\mu(\lambda)$ as $\lambda \to \infty$.

Proposition 2.24 (Karamata's Tauberian Theorem)**.** *If μ is a positive measure on $\mathbb{R}_+$ and*

$$\int_{\mathbb{R}_+} e^{-t\lambda}\, d\mu(\lambda) \sim a t^{-\alpha}, \quad \textit{as } t \to 0,$$

for some $a \in \mathbb{R}$, $\alpha \in (0, \infty)$, then

$$\int_0^l d\mu(\lambda) \sim \frac{a}{\Gamma(\alpha+1)} l^\alpha, \quad \textit{as } l \to \infty.$$

Proof. This rather slick proof comes from [Tay96b], Chapter 8, Prop. 3.2. Define the measure $d\nu(\lambda) = \lambda^{\alpha-1}\, d\lambda$, and for $t \in (0, \infty)$ set $d\mu_t(\lambda) = t^\alpha d\mu(\lambda/t)$. Notice that $d\nu_t(\lambda) \equiv d\nu(\lambda)$ is independent of t. The hypothesis is that $\lim_{t\to 0} t^\alpha \int_0^\infty e^{-t\lambda}\, d\mu(\lambda) = a$, which can be written as

$$\lim_{t\to 0} \int_0^\infty e^{-\lambda}\, d\mu_t(\lambda) = \frac{a}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda}\, d\nu(\lambda), \tag{2.42}$$

since $\Gamma(\alpha) = \int_0^\infty e^{-\lambda} \lambda^{\alpha-1}\, d\lambda = \int_0^\infty e^{-\lambda}\, d\nu(\lambda)$.

The desired conclusion is that $\lim_{l\to\infty} l^{-\alpha} \int_0^l d\mu(\lambda) = \frac{a}{\Gamma(\alpha+1)} = \frac{a}{\alpha\Gamma(\alpha)}$ which, if we set $l = t^{-1}$ can be written as

$$\lim_{t\to 0} \int_0^1 d\mu_t(\lambda) = \frac{a}{\Gamma(\alpha)} \int_0^1 d\nu(\lambda)$$

since $\alpha^{-1} = \int_0^1 \lambda^{\alpha-1}\, d\lambda$.

Thus the goal is to show

$$\lim_{t\to 0} \int_{\mathbb{R}_+} f(\lambda)\, d\mu_t(\lambda) = \frac{a}{\Gamma(\alpha)} \int_{\mathbb{R}_+} f(\lambda)\, d\nu(\lambda) \tag{2.43}$$

holds for $f(\lambda) = \chi_{[0,1]}(\lambda)$, the characteristic function of $[0,1]$, assuming that (2.43) holds for $f(\lambda) = e^{-\lambda}$.

By rescaling $s\lambda \mapsto \lambda$, (and using the invariance $d\nu_t = d\nu$), it follows that (2.43) holds for all $f(\lambda) = e^{-s\lambda}$, $s > 0$ (both sides of (2.43) pick up a factor of $s^{-\alpha}$). This subalgebra separates points, so by the Stone-Weirstrass theorem is dense in $C_0(\mathbb{R}_+)$, the space of continuous functions on $\mathbb{R}_+$ vanishing at infinity.

Since (2.42) implies that the measures $e^{-\lambda} d\mu_t(\lambda)$ are uniformly bounded for $t \in (0,1]$, it follows by density that (2.43) holds for all f of the form $e^{-\lambda} g(\lambda)$, where $g(\lambda) \in C_0(\mathbb{R}_+)$, but since $\chi_{[0,1]}$ can be approximated in measure by functions in this form, the result follows. □

Combining Propositions 2.22 and 2.24, we have proved the following version of Weyl's asymptotic formula.

Theorem 2.25 (Weyl asymptotics)**.** *Let $P \in \mathrm{Diff}^2(M;E)$ be a positive Laplace-type operator, and let $N(l) = \mu([0,l]) = |\{\lambda_j : 0 \le \lambda_j \le l\}|$ be the eigenvalue counting function of P. Then*

$$N(l) = \frac{\sigma_n \operatorname{Vol}(M) \operatorname{Rank}(E)}{(2\pi)^n} l^{n/2} + o(l^{n/2}), \quad l \to \infty, \tag{2.44}$$

where $\sigma_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit ball in $\mathbb{R}^n$.

Remark. We have incorporated part of the constant $(4\pi)^{n/2}\Gamma(n/2+1)$ into σ_n in the statement above. A somewhat more elegant way to express the above is to note that $\sigma_n \operatorname{Vol}(M)$ is equal to the volume of the unit ball bundle in T^*M. Furthermore, if we use the alternate convention of writing eigenvalues of P as $\lambda = \nu^2$, then with respect to the counting function $N_\nu(l) = \big\{\nu : \nu \le l,\ (P - \nu^2) \text{ not invertible}\big\}$, the asymptotic formula may be written

$$N_\nu(l) = (2\pi)^{-n} \operatorname{Rank}(E) \operatorname{Vol}(B_l^*) + o(l^n), \quad B_l^* = \{(x,\xi) : |\xi| \le l\} \subset T^*M.$$

This is an important example of the so-called *quantum-classical correspondence*, saying that something from quantum mechanics, namely the number of energy levels (eigenvalues) of the quantum system associated to the observable P, is related to something from classical mechanics, namely the volume of the corresponding sublevel set of phase space T^*M.

With $P = \Delta \in \mathrm{Diff}^2(M)$ the scalar Laplacian, we obtain one of the most basic inverse spectral results about Riemannian manifolds, namely, if two Riemannian manifolds M_1 and M_2 are isospectral, then they must necessarily have the same volume:

$$\operatorname{Spec}(\Delta_1) = \operatorname{Spec}(\Delta_2) \implies \operatorname{Vol}(M_1) = \operatorname{Vol}(M_2).$$

Our version, (2.44), of Weyl's asymptotic formula is rather weak, in that the only information about the remainder

$$R(l) = N(l) - \frac{\sigma_n}{(2\pi)^n} \operatorname{Vol}(M) \operatorname{Rank}(E)\, l^{n/2}$$

is that $R(l) = o(l^{n/2})$, which is to say that $\lim_{l\to\infty} l^{-n/2} R(l) = 0$. Considerable effort has been put into obtaining an improved estimate on $R(l)$, using more advanced techniques. The best general result is that

$$R(l) = \mathcal{O}(l^{(n-1)/2})$$

which is sharp, as can be seen in the case that $M = \mathbb{S}^n$ is a standard n-sphere. Better estimates are possible when additional conditions on M are imposed: a celebrated result of Duistermaat and Guillemin states that if the set of periodic geodesics on M has measure zero, then $R(l) = o(l^{(n-1)/2})$.

The typical method for obtaining these improved remainders is to replace the heat trace by the **wave trace**, $\operatorname{Tr} e^{it\sqrt{\Delta}}$, where $e^{it\sqrt{\Delta}}$ denotes the unitary fundamental solution to the evolution equation $(-i\partial_t + \sqrt{\Delta})u = 0$. (Composing the operator $-i\partial_t + \sqrt{\Delta}$ with its adjoint gives the wave operator $-\partial_t^2 + \Delta$.) The fundamental solution to a hyperbolic equation such as the wave equation has a much more delicate analytical structure, and involves much more of the global geometry of M, in particular the global behavior of the geodesic flow. For instance, the wave trace $\operatorname{Tr} e^{it\sqrt{\Delta}}$ has singularities for a discrete set of t consisting of the lengths of closed geodesics on M.

Chapter 3

Atiyah-Singer index theorem

3.1 Overview

Our next subject concerns computation of the index of elliptic operators. Recall from §1.3.3 that an elliptic operator $D \in \Psi^s(M; E^0, E^1)$, $s > 0$ is **Fredholm**; in particular

$$\dim \mathrm{Null}(D),\ \dim \mathrm{Null}(D^*) < \infty.$$

Unfortunately, computing $\dim \mathrm{Null}(D)$ is quite difficult in principle, as it depends sensitively on all the information in D; for instance, a small perturbation of D (even by a lower order term) may cause $\dim \mathrm{Null}(D)$ to change quite drastically.

Definition 3.1. The **index** of a Fredholm operator D is the quantity

$$\mathrm{ind}(D) = \dim \mathrm{Null}(D) - \dim \mathrm{Null}(D^*) \in \mathbb{Z}.$$

The remarkable property of the index is that it is extremely stable with respect to perturbations of D. In fact the index is a *homotopy invariant*, meaning that if D_t, $t \in [a, b]$ is a continuous one-parameter family of Fredholm operators, then $\mathrm{ind}(D_t)$ is constant.

This leads us to the question of computing $\mathrm{ind}(D)$ effectively. In the case that $D \in \mathrm{Diff}^1(M; E^0, E^1)$ is an elliptic operator such that D^*D and DD^* are Laplace operators (in this case we say D is a *Dirac operator*, see more on this below), we can extract the index from a sufficient understanding of the heat kernels e^{-tD^*D} and e^{-tDD^*}.

To motivate this, observe that $\mathrm{Null}(D) = \mathrm{Null}(D^*D)$ by the identity $(D^*Du, u) = \|Du\|^2$. Likewise $\mathrm{Null}(D^*) = \mathrm{Null}(DD^*)$. As observed in Proposition 2.23, the long time behavior of the heat trace $\mathrm{Tr}\, e^{-tD^*D}$ is dominated by the limit

$$\lim_{t\to\infty} \mathrm{Tr}\, e^{-tD^*D} = \dim \mathrm{Null}(D^*D) = \dim \mathrm{Null}(D).$$

In particular, we have

$$\mathrm{ind}(D) = \lim_{t\to\infty} \big(\mathrm{Tr}\, e^{-tD^*D} - \mathrm{Tr}\, e^{-tDD^*} \big).$$

However, as pointed out by McKean and Singer, the difference $\operatorname{Tr} e^{-tD^*D} - \operatorname{Tr} e^{-tDD^*}$ is actually *constant*. One way to see this is to observe that

$$De^{-tD^*D} = e^{-tDD^*}D,$$

which follows by applying D to the equation $(\partial_t + D^*D)u = 0$, $u(0) = u_0$ and using uniqueness of solutions. Then we may compute

$$\begin{aligned}\partial_t \operatorname{Tr}\left(e^{-tD^*D} - e^{-tDD^*}\right) &= \operatorname{Tr}\left(D^*De^{-tD^*D} - DD^*e^{-tDD^*}\right)\\ &= \operatorname{Tr}\left(D^*e^{-tDD^*}D - DD^*e^{-tDD^*}\right)\\ &= \operatorname{Tr}\left([D^*e^{-tDD^*}, D]\right) = 0\end{aligned}$$

using the fact that the trace of a commutator involving smoothing operators vanishes (see the remark following Proposition 2.18).

Alternatively, one can show that D^*D and DD^* have the same strictly positive spectrum with isomorphic eigenspaces. Then from the spectral representation

$$\operatorname{Tr} e^{-tD^*D} = \sum_{\lambda_j \in \operatorname{Spec}(D^*D)} e^{-t\lambda_j} \dim \operatorname{Null}(D^*D - \lambda_j),$$

and similarly for $\operatorname{Tr} e^{-tDD^*}$, it follows (formally at least), that

$$\begin{aligned}\operatorname{Tr} e^{-tD^*D} - \operatorname{Tr} e^{-tDD^*} &= \operatorname{ind}(D) + \sum_{\lambda_j > 0} e^{-t\lambda_j}\big(\dim \operatorname{Null}(D^*D - \lambda_j) - \dim \operatorname{Null}(DD^* - \lambda_j)\big)\\ &= \operatorname{ind}(D).\end{aligned}$$

Exercise 3.1. Show that D^*D and DD^* have the same positive spectrum and that D is an isomorphism from the eigenspace $\operatorname{Null}(D^*D - \lambda)$ to the eigenspace $\operatorname{Null}(DD^* - \lambda)$ if $\lambda > 0$. (Hint: Apply D to the equation $(D^*D - \lambda)u = 0$, then apply D^*.)

In any case, we obtain the **McKean-Singer formula**

$$\operatorname{ind}(D) = \operatorname{Tr}\left(e^{-tD^*D} - e^{-tDD^*}\right), \quad \forall\, t \in \mathbb{R}_+, \tag{3.1}$$

which raises the possibility of using the short time heat asymptotics as $t \searrow 0$ to compute $\operatorname{ind}(D)$. In fact, by the magic of $\mathbb{Z}_2$ grading, we can make the difference of traces in (3.1) appear more like the trace of a single heat kernel. Indeed, if we combine the bundles E^0 and E^1 into the single $\mathbb{Z}_2$ graded bundle $E = E^0 \oplus E^1$, then D and D^* can be combined into the single operator

$$\widetilde{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

acting on the space of sections

$$L^2(M; E) = L^2(M; E^0) \oplus L^2(M; E^1),$$

which obtains a $\mathbb{Z}_2$ grading from the one on E. The operator $\widetilde{D}$ is self-adjoint, and

$$\widetilde{D}^2 = \begin{pmatrix} D^*D & 0 \\ 0 & DD^* \end{pmatrix}.$$

In the $\mathbb{Z}_2$ graded formalism, the trace of an endomorphism $A = \begin{pmatrix} A_{00} & A_{10} \\ A_{01} & A_{11} \end{pmatrix} \in \mathrm{End}(V_0 \oplus V_1)$ is replaced by the **supertrace** (see §3.2.6)

$$\mathrm{Str}\, A = \mathrm{Tr}\, A_{00} - \mathrm{Tr}\, A_{11},$$

and the infinite dimensional analogue leads to the formula

$$\mathrm{ind}(D) = \mathrm{Str}\, e^{-t\widetilde{D}^2} = \int_{M_{\mathrm{diag}}} \mathrm{str}\, H(t,x,x) \mathrm{dVol}_g(x), \quad \forall\ t \in \mathbb{R}_+, \tag{3.2}$$

where $\mathrm{str} : C^\infty\big(M_{\mathrm{diag}}, \mathrm{End}(E)\big) \to C^\infty(M)$ is the fiberwise supertrace on the bundle $\mathrm{End}(E)$.

Note that $\widetilde{D}^2$ is a Laplace-type operator, so by the results of Chapter 2, the Schwartz kernel of $e^{-t\widetilde{D}^2}$ may be considered on the heat space M^2_H, where it has a complete asymptotic expansion at hf, and just as in Proposition 2.22 for the heat trace, we obtain an asymptotic expansion

$$\mathrm{Str}\, e^{-t\widetilde{D}^2} \sim t^{-n/2} \sum_{k=0}^{\infty} b_k t^{k/2},$$

with the b_k obtained in the same way as the a_k in (2.40), except that the fiberwise trace is replaced by the supertrace. Unfortunately, as indicated by the constancy of $\mathrm{Str}\, e^{-t\widetilde{D}^2}$, all of the coefficients b_k for $k < n$ actually *vanish*. For instance,

$$b_0 = (4\pi)^{-n/2}\, \mathrm{Vol}(M)\, \mathrm{str}(I_E) = (4\pi)^{-n/2}\, \mathrm{Vol}(M)\big(\mathrm{Rank}(E_0) - \mathrm{Rank}(E_1)\big) = 0,$$

since $\mathrm{Rank}(E_0)$ must equal $\mathrm{Rank}(E_1)$ ellipticity of D.

Thus, to use the approach of Chapter 2, we would in principle have to compute the coefficients in asymptotic expansion of $H(t,x,y)$ at $\mathsf{hf} \subset M^2_H$ to high order, which is a difficult prospect. Fortunately, there is a very clever trick due to Ezra Getzler[1] wherein the bundle $\mathrm{End}(E)$ is *rescaled* with respect to t (with various components rescaled by different degrees), after which the coefficient of interest in the supertrace expansion of the heat kernel becomes the leading one. The cost of doing this is that the model operator over hf is changed from something like $\Delta + \zeta \cdot \partial_\zeta$ to something more closely resembling the quantum harmonic oscillator $\Delta + |\zeta|^2$. Nevertheless, there is still an explicit solution to the new model problem given by *Mehler's formula* (see §3.3.3 below).

Once again our approach follows [Mel93] (see also [Alb12]), using the heat space to analyze the kernel of $e^{-t\widetilde{D}^2}$. For alternative approaches to the local index formula via the heat supertrace, see [BGV92] and/or [Roe99].

[1] Accomplished when he was an undergraduate, no less!

3.2 Dirac operators

Note that the convention for the Clifford algebra in these notes differs from [Mel93]; instead we follow the conventions of the excellent book [LM89], which serves as a reference for this entire section.

We will be interested in self-adjoint, first order differential operators $D \in \mathrm{Diff}^1(M;E)$ such that D^2 is a Laplace-type operator[2]. In particular, this means that the principal symbol satisfies

$$\sigma(D^2)(x,\xi) = \sigma(D)(x,\xi)^2 = |\xi|^2\, I_E, \quad \forall\, \xi \in T_x^*M.$$

Let us consider what this means algebraically. Fixing $x \in M$ for the moment, and writing $V = T_x^*M$, we may define a map[3]

$$c : V \to \mathrm{End}(E_x), \quad \xi \mapsto -i\sigma(D)(x,\xi), \text{ such that}$$
$$c(\xi)^2 = -|\xi|^2\, I$$

It will be worth our while to consider the algebraic implications of this identity, which says that c induces a representation of the *Clifford algebra* of V on E_x.

3.2.1 Clifford algebras and representations

Definition 3.2. Let $(V, \langle\cdot,\cdot\rangle)$ be a real vector space with a nondegenerate bilinear form. The **Clifford algebra** $\mathrm{C}\ell(V) = \mathrm{C}\ell(V,\langle\cdot,\cdot\rangle)$ is the associative algebra defined by the quotient

$$\mathrm{C}\ell(V,\langle\cdot,\cdot\rangle) := T(V)/\mathcal{I}, \quad T(V) = \bigoplus_{j=0}^{\infty} V^{\otimes j},$$
$$\mathcal{I} = \left\langle v\otimes v + |v|^2\, 1\right\rangle = \langle v\otimes w + w\otimes v + 2\langle v,w\rangle\, 1\rangle$$

of the tensor algebra of V by the ideal generated by elements of the form $v \otimes v + |v|^2\ 1$, where $1 \in V^{\otimes 0} \equiv \mathbb{R}$ is the unit.

Then $\mathrm{C}\ell(V)$ is the universal algebra generated by V and satisfying $v^2 = -|v|^2$ (equivalently $vw + wv = -2\langle v,w\rangle$), meaning that any linear map ϕ from V to an algebra A whose image satisfies this relation extends to a unique map $\widetilde{\phi} : \mathrm{C}\ell(V) \to A$.

We denote the complexified Clifford algebra by $\mathbb{C}\ell(V) = \mathrm{C}\ell(V)\otimes_{\mathbb{R}}\mathbb{C}$. This is equivalent to the Clifford algebra over $\mathbb{C}$ of the complex vector space $V\otimes_{\mathbb{R}}\mathbb{C}$, with the quadratic form extended complex linearly[4].

The Clifford algebra is defined for a bilinear form of any signature (see [LM89]), though we shall be solely concerned with the case that it is positive definite, which we assume from

[2] Such as $\widetilde{D}$ in the previous section, but we will drop the tilde and denote this simply as D from now on.

[3] The introduction of the somewhat strange seeming sign $-i$ in the formula is a choice which leads to the standard convention for the Clifford algebra below. In fact, you can think of it as undoing the i way back in the definition (1.3) of the principal symbol.

[4] Note that the quadratic form on $V\otimes\mathbb{C}$ is related to a *complex bilinear*, as opposed to a *Hermitian* form.

now on. (Note that there is no such thing as signature in the complex case, a fact which considerably simplifies the theory of complex Clifford algebras.)

In practice it is easiest to work with an orthonormal basis $\{e_1, \ldots, e_n\}$ for V, and then $\mathrm{C}\ell(V)$ is generated as an algebra over $\mathbb{R}$ by $\{1, e_1, \ldots, e_n\}$, subject to the relations

$$e_i^2 = -1 \quad \text{and} \quad e_i e_j + e_j e_i = 0,\ i \neq j. \tag{3.3}$$

From this it is clear that $\mathrm{C}\ell(V)$ has a basis given by

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} : i_1 < i_2 < \cdots < i_k,\ 0 \leq k \leq n\},$$

and therefore $\dim \mathrm{C}\ell(V) = 2^n$. Note that there is a natural embedding $V \hookrightarrow \mathrm{C}\ell(V)$ given by the span of those basis elements with $k = 1$. In fact, there is a natural filtration on $\mathrm{C}\ell(V)$:

$$\mathbb{R} = \mathrm{C}\ell^{(0)}(V) \subset \mathrm{C}\ell^{(1)}(V) \subset \cdots \subset \mathrm{C}\ell^{(n)}(V) = \mathrm{C}\ell(V), \quad \mathrm{C}\ell^{(j)}(V) \cdot \mathrm{C}\ell^{(k)}(V) \subset \mathrm{C}\ell^{(j+k)}(V),$$

where $\mathrm{C}\ell^{(k)}(V)$ is the linear span of those basis elements $e_{i_1} \cdots e_{i_l}$ with $0 \leq l \leq k$ (or more invariantly, the image of the filtration $\sum_{l=0}^{k} V^{\otimes l}$ of $T(V)$ in the quotient). Note that we are forced to include elements with length less than k in order that the filtration is compatible with multiplication. There are short exact sequences

$$\mathrm{C}\ell^{(k-1)}(V) \hookrightarrow \mathrm{C}\ell^{(k)}(V) \to \Lambda^k V$$

for each k, which taken together form a natural isomorphism of vector spaces (not of algebras!)

$$\mathrm{C}\ell(V) \overset{\cong}{\to} \Lambda V = \bigoplus_{j=0}^{n} \Lambda^n V. \tag{3.4}$$

The inverse map $\Lambda V \to \mathrm{C}\ell(V)$, can be written in terms of a basis by sending $e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto e_{i_1} \cdots e_{i_k}$. We will sometimes write $\Lambda^k V \subset \mathrm{C}\ell(V)$ for the inverse image of $\Lambda^k V$ with respect to the isomorphism above.

Even though $\mathrm{C}\ell(V)$ is not graded as an algebra by $\mathbb{Z}$, it does admit a $\mathbb{Z}_2$ grading. Indeed,

$$\begin{gathered}\mathrm{C}\ell(V) = \mathrm{C}\ell^0(V) \oplus \mathrm{C}\ell^1(V), \quad \mathrm{C}\ell^i(V) \cdot \mathrm{C}\ell^j(V) \subset \mathrm{C}\ell^{i+j (\mathrm{mod}\ 2)}(V), \\ \mathrm{C}\ell^0(V) = \operatorname{span}\{e_{i_1} \cdots e_{i_k} : k \text{ even}\}, \quad \mathrm{C}\ell^1(V) = \operatorname{span}\{e_{i_1} \cdots e_{i_k} : k \text{ odd}\}.\end{gathered}$$

(Please note the notational distinction between the components $\mathrm{C}\ell^{(j)}(V)$ of the filtration and the summands $\mathrm{C}\ell^j(V)$ of the $\mathbb{Z}_2$ grading!) Alternatively, the involution $\alpha \in \mathrm{Aut}(V)$ given by $\alpha(v) = -v$ can be extended to an involution on $\mathrm{C}\ell(V)$ and then $\mathrm{C}\ell^0(V)$ and $\mathrm{C}\ell^1(V)$ are its $+1$ and -1 eigenspaces, respectively. Note that $V = \mathrm{C}\ell^1(V) \cap \mathrm{C}\ell^{(1)}(V)$.

For each n, we let $\mathrm{C}\ell_n = \mathrm{C}\ell(\mathbb{R}^n)$ denote the Clifford algebra of Euclidean n-space with the standard inner product, and $\mathbb{C}\ell_n$ its complexification. A choice of orthonormal basis $\{e_1, \ldots, e_n\}$ for (V, q) identifies $\mathrm{C}\ell(V)$ with $\mathrm{C}\ell_n$ and $\mathbb{C}\ell(V)$ with $\mathbb{C}\ell_n$. Since we will be primarily interested in complex representations below, we will focus on the complex algebras $\mathbb{C}\ell_n$.

Consider the low dimensional cases. In the first case, $\mathbb{C}\ell_1$ is generated by $\{1, e_1\}$, and we have

$$\mathbb{C}\ell_1 \cong \mathbb{C} \oplus \mathbb{C} \tag{3.5}$$

as an algebra, under the identification $1 \mapsto (1,1)$, $e_1 \mapsto (i, -i)$. (The $\mathbb{Z}_2$ grading on $\mathbb{C} \oplus \mathbb{C}$ is the splitting into vectors of the form (a, a) and those of the form $(a, -a)$).

Next, $\mathbb{C}\ell_2$ is generated as a vector space by $\{1, e_1, e_2, e_1e_2\}$, with multiplication determined by the relations (3.3). This can be identified with the algebra $\mathrm{Mat}_2(\mathbb{C})$ of 2×2 complex matrices. Indeed, a basis for the latter space is given by the classical *Pauli matrices*[5]

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \gamma_1\gamma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{3.6}$$

and these are easily seen to satisfy the same relations as $\{1, e_1, e_2, e_1e_2\}$, so $1 \mapsto \sigma_0$, $e_i \mapsto \sigma_i$, $i = 1, 2$ defines an isomorphism

$$\mathbb{C}\ell_2 \cong \mathrm{Mat}_2(\mathbb{C}). \tag{3.7}$$

The fundamental *periodicity result* is the following.

Proposition 3.3. *For each n, there is an isomorphism*

$$\mathbb{C}\ell_n \cong \mathbb{C}\ell_{n-2} \otimes \mathbb{C}\ell_2.$$

In particular,

$$\mathbb{C}\ell_{2n} \cong \mathrm{Mat}_{2^n}(\mathbb{C}), \quad \mathbb{C}\ell_{2n+1} \cong \mathrm{Mat}_{2^n}(\mathbb{C}) \oplus \mathrm{Mat}_{2^n}(\mathbb{C}). \tag{3.8}$$

Proof. First we construct a map $f : \mathbb{C}^n \to \mathbb{C}\ell_{n-2} \otimes \mathbb{C}\ell_2$ satisfying the Clifford relations. This is accomplished by defining

$$\begin{aligned} f(e_j) &= i\, e_j \otimes e_1e_2, \quad 1 \le j \le n-2, \\ f(e_{n-1}) &= 1 \otimes e_1, \quad f(e_n) = 1 \otimes e_2. \end{aligned}$$

Then it is easy to check that $f(e_j)f(e_k) + f(e_k)f(e_j) = -2\delta_{jk}(1 \otimes 1)$, so by the universal property, f extends to a homomorphism $\widetilde{f} : \mathbb{C}\ell_n \to \mathbb{C}\ell_{n-2} \otimes \mathbb{C}\ell_2$. This is easily seen to map onto a set of generators, so $\widetilde{f}$ is surjective, and since the dimensions of both spaces are the same, $\widetilde{f}$ is an isomorphism. The identifications (3.8) follows by induction from the base cases (3.5) and (3.7). $\square$

Remark. The periodicity of $\mathbb{C}\ell_n$ is intimately related to Bott periodicity in complex topological K-theory. The real Clifford algebras $\mathrm{C}\ell_n$ satisfy the 8-periodic identity $\mathrm{C}\ell_n \cong \mathrm{C}\ell_{n-8} \otimes \mathrm{C}\ell_8$ (though the proof requires consideration of Clifford algebras with quadratic forms of arbitrary signature), which is in turn related to the 8-fold periodicity of real topological K-theory.

[5] listed here possibly in a nonstandard order with nonstandard signs

According to the result, $\mathbb{C}\ell_{2n+1} \cong \mathrm{Mat}_{2^n}(\mathbb{C}) \oplus \mathrm{Mat}_{2^n}(\mathbb{C})$ contains two copies of $\mathbb{C}\ell_{2n} \cong \mathrm{Mat}_{2^n}(\mathbb{C})$, and one might suppose that these are related to the $\mathbb{Z}_2$ grading. Indeed, in the other direction, under the identification of $\mathrm{Mat}_{2^n}(\mathbb{C})$ with $\mathbb{C}\ell_{2n}$, the $\mathbb{Z}_2$ grading $\mathbb{C}\ell^0_{2n} \oplus \mathbb{C}\ell^1_{2n}$ becomes the grading $\mathrm{Mat}^0_{2^n}(\mathbb{C}) \oplus \mathrm{Mat}^1_{2^n}(\mathbb{C})$ where $\mathrm{Mat}^0_{2^n}(\mathbb{C})$ consists of block diagonal matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\mathrm{Mat}^1_{2^n}(\mathbb{C})$ consists of off-diagonal matrices $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ (this follows from (3.6) and induction). Identifying these with $A \oplus B$ shows that $\mathbb{C}\ell_{2n}$ contains two copies of $\mathbb{C}\ell_{2n-1}$, one for each component of the $\mathbb{Z}_2$ grading. Directly at the level of Clifford algebras, this is the following result.

Proposition 3.4. *For each n, there is an isomorphism*

$$\mathbb{C}\ell_n \cong \mathbb{C}\ell^0_{n+1}.$$

Proof. As in the previous proof, we define a map $\mathbb{C}^n \to \mathbb{C}\ell^0_{n+1}$ by

$$f(e_j) = e_j e_{n+1}, \quad 1 \le j \le n.$$

This satisfies $f(e_j)^2 = e_j e_{n+1} e_j e_{n+1} = -e_j^2 e_{n+1}^2 = -1$ so extends to a homomorphism $\widetilde{f} : \mathbb{C}\ell_n \to \mathbb{C}\ell^0_{n+1}$. It is easy to see that the image of $\widetilde{f}$ contains all elements of the form $e_{i_1} \cdots e_{i_{2k}}$, so $\widetilde{f}$ is surjective, and therefore an isomorphism by dimensionality. $\square$

Definition 3.5. A **representation** of $\mathrm{C}\ell(V)$ is a (finite dimensional) real vector space E along with an algebra homomorphism $\rho : \mathrm{C}\ell(V) \to \mathrm{End}(E)$. We say E is **reducible** if $E = E' \oplus E''$ with nontrivial summands, with $\rho = \rho' \oplus \rho'' : \mathrm{C}\ell(V) \to \mathrm{End}(E') \oplus \mathrm{End}(E'')$. Otherwise E is **irreducible**. By finite dimensionality, every representation is a finite direct sum of irreducible ones.

We say E is a **graded representation** if $E = E^0 \oplus E^1$ and

$$\mathrm{C}\ell^j(V) : E^k \to E^{j+k (\mathrm{mod}\ 2)}$$

Equivalently, ρ is a homomorphism of graded algebras, meaning $\rho : \mathrm{C}\ell^j(V) \to \mathrm{End}^j(E)$, where $\mathrm{End}^0(E) = \mathrm{Hom}(E^0, E^0) \oplus \mathrm{Hom}(E^1, E^1)$ and $\mathrm{End}^1(E) = \mathrm{Hom}(E^0, E^1) \oplus \mathrm{Hom}(E^1, E^0)$.

We say (E, ρ) is a **complex representation** if $\dim_{\mathbb{R}}(E)$ is even and there is a complex structure $J \in \mathrm{Aut}(E)$ such that $J^2 = -I$ and which commutes with $\rho(a)$ for all $a \in \mathrm{C}\ell(V)$. In this case E obtains the structure of a complex vector space and ρ extends to a homomorphism $\rho : \mathbb{C}\ell(V) \to \mathrm{End}_{\mathbb{C}}(E)$. We consider only complex representations below.

Example 3.6. The most obvious example of a nontrivial representation of $\mathrm{C}\ell(V)$ is its action on itself by left multiplication. Using (3.4), we can regard ΛV as a representation, and it is easy to check that the action is generated by

$$\begin{aligned} \rho &: \mathrm{C}\ell(V) \to \mathrm{End}(\Lambda V), \\ \rho(v) &= v \wedge \cdot - v \lrcorner \cdot, \quad v \in V. \end{aligned} \tag{3.9}$$

Notice that this is exactly the action defined by the principal symbol of the Hodge-de Rham operator $d+d^*$ on forms (c.f. examples 1.7, 1.10 and 1.20).

This representation is also graded with respect to $\Lambda V = \Lambda^{\text{even}}V \oplus \Lambda^{\text{odd}}V$, which is of course induced by the identification with $\mathrm{C}\ell(V) = \mathrm{C}\ell^0(V) \oplus \mathrm{C}\ell^1(V)$.

Propositions 3.3 and 3.4 lead immediately to a complete classification of the complex irreducible representations of Clifford algebras, since $\mathrm{Mat}_n(\mathbb{C})$ has a unique irreducible representation on $\mathbb{C}^n$ (it is a so-called *simple algebra*), and $\mathrm{Mat}_n(\mathbb{C}) \oplus \mathrm{Mat}_n(\mathbb{C})$ has two distinct irreducible representations on $\mathbb{C}^n$ given by projection to one or the other factors.

Proposition 3.7. *The Clifford algebra $\mathbb{C}\ell_{2n}$ has a unique irreducible representation S_{2n} with $\dim_{\mathbb{C}}(\mathrm{S}_{2n}) = 2^n$. The Clifford algebra $\mathbb{C}\ell_{2n+1}$ has two inequivalent irreducible representations S^+_{2n+1} and S^-_{2n+1}, each of dimension 2^n.*

Recalling that the $\mathbb{Z}_2$ grading on $\mathrm{Mat}_{2^n}(\mathbb{C}) \cong \mathbb{C}\ell_{2n}$ is into block diagonal and off-diagonal $2^{n-1} \times 2^{n-1}$ matrices, it follows by writing $\mathbb{C}^{2^n} = \mathbb{C}^{2^{n-1}} \oplus \mathbb{C}^{2^{n-1}}$ that the fundamental representation S_{2n} of $\mathbb{C}\ell_{2n}$ is a graded representation. More abstractly, Proposition 3.4 leads to the following result.

Proposition 3.8. *There is an equivalence of categories between graded representations of $\mathbb{C}\ell_n$ and (ungraded) representations of $\mathbb{C}\ell_{n-1}$.*

Proof. In one direction, if $E = E^0 \oplus E^1$ is a graded representation of $\mathbb{C}\ell_n$, then E^0 is a $\mathbb{C}\ell_{n-1} \cong \mathbb{C}\ell^0_n$ representation. (Note that E^1 is also a representation, possibly a different one.) Conversely, given a representation F of $\mathbb{C}\ell_{n-1}$, we can set $E = \mathbb{C}\ell_n \otimes_{\mathbb{C}\ell^0_n} F$, which makes E into a graded $\mathbb{C}\ell_n$ representation since $\mathbb{C}\ell^0_n \oplus \mathbb{C}\ell^1_n$ is a graded representation of $\mathbb{C}\ell_n$ with respect to left multiplication. $\square$

It follows that the unique irreducible representation S_{2n} of $\mathbb{C}\ell_{2n}$ splits as $\mathrm{S}_{2n} = \mathrm{S}^+_{2n} \oplus \mathrm{S}^-_{2n}$ with S^+_{2n} and S^-_{2n} the two inequivalent $\mathbb{C}\ell_{2n-1}$ representations. Thus S_{2n} has two *inequivalent* gradings, either $\mathrm{S}^0 \oplus \mathrm{S}^1 = \mathrm{S}^+ \oplus \mathrm{S}^-$, or $\mathrm{S}^0 \oplus \mathrm{S}^1 = \mathrm{S}^- \oplus \mathrm{S}^+$. On the other hand, the unique irreducible graded representation of $\mathbb{C}\ell_{2n+1}$ has the form $\mathrm{S}^0_{2n+1} \oplus \mathrm{S}^1_{2n+1}$, with each factor isomorphic to S_{2n}.

There is a nifty way to distinguish between these representations. In $\mathbb{C}\ell_n$, define the **volume element** by

$$\omega_n = i^p e_1 \cdots e_n, \quad p = \begin{cases} n/2, & n \text{ even}, \\ (n+1)/2, & n \text{ odd}. \end{cases} \tag{3.10}$$

Then it is straightforward to check that

$$\omega_n^2 = 1, \quad \text{and} \quad e_j \omega_n = (-1)^{n+1} \omega_n e_j \ \forall\, j.$$

In particular, in odd dimensions, ω_{2n+1} is central, and the two distinct irreducible representations $\mathrm{S}^\pm_{2n+1}$ are distinguished by $\rho(\omega_{2n+1}) = \pm 1$.

In even dimensions, the splitting $\mathrm{S}_{2n} = \mathrm{S}_{2n}^+ \oplus \mathrm{S}_{2n}^-$ may be defined according to the ± 1 eigenspaces of ω_{2n}. Since the e_j anti-commute with ω_{2n} in this case, it follows that elements of $\mathbb{C}\ell_{2n}^0$ commute with ω_{2n} while elements of $\mathbb{C}\ell_{2n}^1$ anti-commute with it, so this is a graded representation. Furthermore, since $\omega_{2n} \equiv \omega_{2n-1}e_{2n}$ is consistent with the identification $\mathbb{C}\ell_{2n-1} \cong \mathbb{C}\ell_{2n}^0$, it follows that $\mathrm{S}_{2n}^\pm \cong \mathrm{S}_{2n-1}^\pm$.

Definition 3.9. We refer to $\mathrm{S}_{2n} = \mathrm{S}_{2n}^+ \oplus \mathrm{S}_{2n}^-$ as the (graded) **spinor** representation of $\mathbb{C}\ell_{2n}$. The representations $\mathrm{S}_{2n}^\pm = \mathrm{S}_{2n-1}^\pm$ are called the **half-spinor** representations of $\mathbb{C}\ell_{2n}^0 \cong \mathbb{C}\ell_{2n-1}$.

Remark. As described above, the spinor representation is defined via the isomorphisms $\mathbb{C}\ell_{2n} \cong \mathrm{Mat}_{2^n}(\mathbb{C})$. For a direct construction, consider $\Lambda\mathbb{C}^n = \bigoplus_k \Lambda_{\mathbb{C}}^k\mathbb{C}^n$. The standard Hermitian inner product $(z, w) = z\overline{w}$ extends to one on $\Lambda\mathbb{C}^n$ and defines an interior product

$$v \lrcorner \cdot : \Lambda_{\mathbb{C}}^k\mathbb{C}^n \to \Lambda_{\mathbb{C}}^{k-1}\mathbb{C}^n, \quad v \in \mathbb{C}^n$$

as the adjoint to the operation $v \wedge \cdot \in \mathrm{Hom}(\Lambda_{\mathbb{C}}^{k-1}\mathbb{C}^n, \Lambda_{\mathbb{C}}^k\mathbb{C}^n)$. Then

$$\mathbb{R}^{2n} \cong \mathbb{C} \ni v \mapsto v \wedge \cdot - v \lrcorner \cdot \in \mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbb{C}}\mathbb{C}^n)$$

defines an $\mathbb{R}$-linear[6] map satisfying the Clifford relations, hence extends to a nontrivial complex representation of $\mathrm{C}\ell_{2n}$, which by reason of dimension must be the unique irreducible representation S_{2n}.

Note that this is not the same as the complexification of the representation of Example 3.6, as the latter has complex dimension 2^{2n} rather than 2^n.

3.2.2 Dirac operators on a manifold

We now transfer the theory of Clifford algebras to the tangent space of a manifold. Recall that the tangent bundle $TM \to M$ of an oriented Riemannian manifold is associated to a **principal** $\mathrm{SO}(n)$ **bundle**

$$P_{\mathrm{SO}} \to M, \tag{3.11}$$

meaning P_{SO} is a locally trivial fiber bundle whose fibers are principal $\mathrm{SO}(n)$ spaces; equivalently P_{SO} carries a free right $\mathrm{SO}(n)$ action whose quotient is precisely the map (3.11). Explicitly, P_{SO} may be realized as the (orthogonal) **frame bundle**

$$P_{\mathrm{SO}} = \{\phi \in \mathrm{Iso}(\mathbb{R}^n, T_xM) : x \in M\} \tag{3.12}$$

consisting of oriented orthogonal isomorphisms from $(\mathbb{R}^n, g_{\mathrm{Euc}})$ into the tangent spaces (T_xM, g_x), with the right action by $\mathrm{SO}(n)$ given by precomposition: $\mathrm{SO}(n) \ni a : \phi \mapsto \phi \circ a$.

The tangent bundle TM may be recovered from P_{SO} by the **associated bundle** construction:

$$TM = P_{\mathrm{SO}} \times_{\rho_n} \mathbb{R}^n := (P_{\mathrm{SO}} \times \mathbb{R}^n)/\mathrm{SO}(n) \to M,$$

[6]Note that the map is *not* $\mathbb{C}$-linear since the Hermitian inner product is skew-linear in the second variable!

where we take the (left) $\mathrm{SO}(n)$ action on the product $P_{\mathrm{SO}} \times \mathbb{R}^n$ by

$$a \cdot (\phi, v) := (\phi\, a^{-1}, \rho_n(a)\, v),$$

and where $\rho_n : \mathrm{SO}(n) \to \mathrm{GL}(\mathbb{R}^n)$ is the standard representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$. In other words, the quotient TM is the equivalence classes of pairs $(\phi, v) \sim (\phi\, a^{-1}, \rho_n(a)v)$, which in terms of (3.12) is realized explicitly by evaluation: $T_xM \ni \xi = \phi(v)$.

Generally speaking, given any linear representation $\rho : \mathrm{SO}(n) \to \mathrm{GL}(V)$, there is an associated vector bundle

$$E = P_{\mathrm{SO}} \times_\rho V = \big\{[(\phi, v)] : (\phi, v) \sim (\phi\, a^{-1}, \rho(a)v)\big\} \to M,$$

whose fibers are isomorphic to V. All of the standard geometric bundles are associated bundles with respect to various representations:

$$\begin{gathered} T^*M = P_{\mathrm{SO}} \times_{\rho_n^*} \mathbb{R}^n, \quad \Lambda^k M = P_{\mathrm{SO}} \times_{\Lambda^k \rho_n^*} \Lambda^k \mathbb{R}^n, \\ \bigotimes^r TM = P_{\mathrm{SO}} \times_{\bigotimes^r \rho_n} \bigotimes^r \mathbb{R}^n \quad \text{etc.} \end{gathered} \tag{3.13}$$

(Here $\rho_n^* = (\rho_n^{-1})^\dagger$ denotes the dual, or contragredient, representation.)

Definition 3.10. Let M be an oriented Riemannian manifold. The **Clifford bundle** $\mathrm{C}\ell(M)$ is the associated bundle

$$\mathrm{C}\ell(M) = P_{\mathrm{SO}} \times_{\mathrm{c}\ell(\rho_n)} \mathrm{C}\ell(\mathbb{R}^n) \to M$$

where $\mathrm{c}\ell(\rho_n) : \mathrm{SO}(n) \to \mathrm{Aut}\big(\mathrm{C}\ell(\mathbb{R}^n)\big)$ is the action induced by the standard action on $\mathbb{R}^n$. We denote by $\mathbb{C}\ell(M)$ the complexified Clifford bundle $P_{\mathrm{SO}} \times_{\mathrm{c}\ell(\rho_n)} \mathbb{C}\ell(\mathbb{R}^n)$. The fiber of $\mathrm{C}\ell(M)$ (resp. $\mathbb{C}\ell(M)$) at $x \in M$ is just $\mathrm{C}\ell_x(M) = \mathrm{C}\ell(T_xM, g_x)$ (resp. $\mathbb{C}\ell(T_xM, g_x)$).

There is a natural inclusion $TM \hookrightarrow \mathrm{C}\ell(M)$ as a vector subbundle. Alternatively, we may regard T^*M as a subbundle using the identification $TM \cong T^*M$ afforded by the metric. We will use this identification implicitly below, and often fail to distinguish between tangent and cotangent vectors.

We say a vector bundle $E \to M$ is a **Clifford module**, or $\mathrm{C}\ell(M)$**-module**, if there is a multiplicative action

$$\begin{gathered} \mathrm{c}\ell : \mathrm{C}\ell(M) \otimes E \to E, \quad \eta \otimes v \mapsto \mathrm{c}\ell(\eta)v, \\ \mathrm{c}\ell(\eta_1)(\mathrm{c}\ell(\eta_2)v) = \mathrm{c}\ell(\eta_1\eta_2)v. \end{gathered} \tag{3.14}$$

In particular, E_x is a representation of $\mathrm{C}\ell_x(M)$ for all $x \in M$. We are mostly interested in the case that E is a complex bundle with Hermitian inner product, with a complex action $\mathbb{C}\ell(M) \otimes E \to E$, and we assume that the action by unit vectors is unitary:

$$\langle \mathrm{c}\ell(e)v, \mathrm{c}\ell(e)w\rangle = \langle v, w\rangle, \quad |e|^2 = 1.$$

In light of $\mathrm{c}\ell(e)^2 = -1$, this is equivalent to $\langle \mathrm{c}\ell(e)v, w\rangle = -\langle v, \mathrm{c}\ell(e)w\rangle$.

We say E is a graded Clifford module if $E = E^0 \oplus E^1$ and the action (3.14) is graded: $\mathrm{C}\ell^j(M) \otimes E^k \to E^{j+k (\mathrm{mod}\ 2)}$.

Example 3.11. The bundle $\Lambda M = \bigoplus_{j=1}^n \Lambda^j T^*M$ is a Clifford module, with action given as in (3.9). With respect to the grading $\Lambda M = \Lambda^{\text{even}} M \oplus \Lambda^{\text{odd}} M$, it is a graded Clifford module. Taking complexification $\Lambda M \otimes \mathbb{C}$ leads to a $\mathbb{C}\ell(M)$ module.

Recall that a **connection**, or **covariant derivative**, on a vector bundle E is a first order differential operator

$$\nabla : C^\infty(M; E) \to C^\infty(M; T^*M \otimes E)$$

which satisfies the Liebnitz rule:

$$\nabla(fs) = df \otimes s + f\nabla s, \quad s \in C^\infty(M; E),\ f \in C^\infty(M).$$

Fixing a vector field $V \in C^\infty(M; TM)$ and taking the natural contraction between TM and T^*M gives the differential operator

$$\nabla_V : C^\infty(M) \to C^\infty(M), \quad \nabla_V(fs) = V(f)\, s + f\nabla_V s.$$

If E has a Hermitian (resp. real positive definite) inner product, then ∇ is a **unitary (resp. orthogonal) connection** provided

$$d\langle s, t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle \iff V\langle s, t\rangle = \langle \nabla_V s, t\rangle + \langle s, \nabla_V t\rangle.$$

Example 3.12. Recall that if M is equipped with a Riemannian metric, then TM admits a unique orthogonal connection under the requirement that the *torsion* $\nabla_X Y - \nabla_Y X - [X, Y]$ vanishes. The result is the **Levi-Civita connection** $\nabla = \nabla^{\text{LC}}$, which may be defined via the *Koszul formula*

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle \\ &\quad + X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle. \end{aligned} \tag{3.15}$$

Apart from the signs, this formula is easy to remember. The signs are fixed by the orthogonality condition (exchanging Y and Z in the formula and adding should result in $2X\langle Y, Z\rangle$, so the fourth sign is $+$, the first and third signs are the same, and the fifth and sixth signs are opposite) and the torsion free condition (exchanging X and Y in the formula and subtracting should result in $2\langle [X,Y], Z\rangle$, so the first sign is $+$, the second and third signs are opposite, and the fourth and fifth signs are the same). We can encode the Levi-Civita connection in terms of a local frame $\{e_1, \dots, e_n\}$ for TM in terms of the coefficients

$$\Gamma_{kij} = \langle \nabla_{e_i} e_j, e_k\rangle, \quad \text{or} \quad {\Gamma^k}_{ij} \text{ s.t. } \nabla_{e_i} e_j = \sum_k {\Gamma^k}_{ij} e_k,$$

which can be computed by the above. Particularly useful are the cases when $\{e_1, \dots, e_n\}$ are orthogonal, in which case the second row of (3.15) vanishes, or when $\{e_1, \dots, e_n\} = \{\partial_{x_1}, \dots, \partial_{x_n}\}$ is a coordinate basis, in which case we call the Γ_{kij} **Christoffel symbols** and the first row of (3.15) vanishes, giving the classical formula $\Gamma_{kij} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$.

In fact, we recall that the connection can be associated with an object (a principal bundle connection) on the principal bundle P_{SO}, which then induces a connection on any associated bundle. Thus the bundles (3.13) are all equipped with canonical orthogonal connections, which we collectively refer to as the Levi-Civita connection.

In particular, the Clifford bundle itself inherits the Levi-Civita connection

$$\nabla = \nabla^{\mathrm{LC}} : C^\infty\big(M; \mathrm{C}\ell(M)\big) \to C^\infty\big(M; T^*M \otimes \mathrm{C}\ell(M)\big).$$

Definition 3.13. Let $E \to M$ be a Hermitian Clifford module. A unitary connection ∇ on E is a **Clifford connection** if it satisfies the following compatibility condition with respect to the action of $\mathbb{C}\ell(M)$:

$$\nabla\big(\mathrm{c}\ell(\eta)s\big) = \mathrm{c}\ell(\nabla^{\mathrm{LC}}\eta)s + \mathrm{c}\ell(\eta)\nabla s. \tag{3.16}$$

Given such a structure, we define a **Dirac operator** $D \in \mathrm{Diff}^1(M; E)$ by the composite

$$D : C^\infty(M; E) \overset{\nabla}{\to} C^\infty(M; T^*M \otimes E) \overset{\mathrm{c}\ell}{\to} C^\infty(M; E)$$

At a point $x \in M$, D has the local expression

$$(Ds)(x) = \sum_{j=1}^n \mathrm{c}\ell(e_j)\nabla_{e_j} s(x), \quad s \in C^\infty(M; E), \tag{3.17}$$

where $\{e_1, \ldots, e_n\}$ is a frame for TM near T_xM which is orthonormal at T_xM (on the choice of which D does not depend).

Proposition 3.14. *A Dirac operator $D \in \mathrm{Diff}^1(M; E)$ is a formally self-adjoint elliptic operator, with principal symbol*

$$\sigma(D)(x, \xi) = i\mathrm{c}\ell(\xi) \in C^\infty\big(T^*M; \mathrm{End}(E)\big). \tag{3.18}$$

In particular, D^2 has symbol $\sigma(D^2) = |\xi|^2 I$, so D^2 is a positive Laplace-type operator.

Proof. The principal symbol of any connection is easily computed to be

$$\sigma(\nabla)(x, \xi) = i\xi \otimes \cdot \in C^\infty\big(T^*M; \mathrm{Hom}(E, T^*M \otimes E)\big),$$

and composing this with the Clifford action gives (3.18).

To see that D is self-adjoint, we first note that, since ∇ is a unitary connection on E,

$$\langle \nabla_V s, t\rangle = V\langle s, t\rangle - \langle s, \nabla_V t\rangle \in C^\infty(M), \quad s, t \in C^\infty(M; E), \quad V \in C^\infty(M; TM).$$

Integrating this over M, and recalling that $\int_M V(f)\,\mathrm{dVol}_g =: \int_M f\,\mathrm{div}(V)\,\mathrm{dVol}_g$ where $\mathrm{div}(V) \in C^\infty(M)$ is the *divergence* of V, we have the L^2 adjoint formula

$$(\nabla_V s, t)_{L^2} = (s, -\nabla_V t)_{L^2} + (s, \mathrm{div}(V)t)_{L^2}$$

so in other words $\nabla_V^* = -\nabla_V + \operatorname{div}(V)$ for unitary connection. Then we may use the local expression (3.18) to compute

$$\begin{aligned}
\langle Ds, t\rangle = \sum_j \big\langle \mathrm{c}\ell(e_j)\nabla_{e_j} s, t\big\rangle &= -\sum_j \langle s, \nabla_{e_j}^*\big(\mathrm{c}\ell(e_j)t\big)\rangle \\
&= \sum_j \big\langle s, \nabla_{e_j}\big(\mathrm{c}\ell(e_j)t\big)\big\rangle - \langle s, \operatorname{div}(e_j)\mathrm{c}\ell(e_j)t\rangle \\
&= \sum_j \big\langle s, \mathrm{c}\ell(e_j)\nabla_{e_j} t\big\rangle + \langle s, \mathrm{c}\ell(\nabla_{e_j}^{\mathrm{LC}} e_j)t\rangle - \langle s, \operatorname{div}(e_j)\mathrm{c}\ell(e_j)t\rangle. \\
&= \langle s, Dt\rangle + \sum_j \langle s, \mathrm{c}\ell(\nabla_{e_j}^{\mathrm{LC}} e_j)t\rangle - \langle s, \operatorname{div}(e_j)\mathrm{c}\ell(e_j)t\rangle.
\end{aligned}$$

Observe that expression must be independent of the choice of orthonormal frame $\{e_1, \dots, e_n\}$, and we can always arrange to satisfy $\nabla_{e_i} e_j = 0$ at a given point $x \in M$. Since $\operatorname{div}(V) = \sum_j e_j \lrcorner \nabla_{e_j} V$ it follows that the second two terms above vanish at x when computed with such a frame. But since x was arbitrary and the expression is independent of the choice of frame, these additional terms must vanish identically. □

Since $T^*M \subset \mathrm{C}\ell^1(M)$ is in the odd-graded part of $\mathrm{C}\ell(M)$, it follows that if $E = E^0 \oplus E^1$ is a graded Clifford module, then the Dirac operator has the form

$$\begin{gathered}
D = \begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix} : C^\infty(M; E^0 \oplus E^1) \to C^\infty(M; E^0 \oplus E^1), \\
D_0 \in \operatorname{Diff}^1(M; E^0, E^1), \quad D_1 = (D_0)^* \in \operatorname{Diff}^1(M; E^1, E^0).
\end{gathered}$$

Example 3.15. To revisit our one (and only) example so far, taking the Levi-Civita connection on the Clifford module ΛM with Clifford action (3.9) leads to a Dirac operator $D \in \operatorname{Diff}^1(M; \Lambda M)$, which of course is our old friend, the Hodge de Rham operator

$$D = d + d^* : C^\infty(M; \Lambda M) \to C^\infty(M; \Lambda M).$$

With respect to the grading, D_0 and D_1 are both given by $d + d^*$, but considered as operators from even to odd degree forms and vice versa.

One way of obtaining new Dirac operators is the following. Suppose E is a Hermitian Clifford module with Dirac operator $D = \mathrm{c}\ell \circ \nabla^E \in \operatorname{Diff}^1(M; E)$, and suppose F is any other Hermitian bundle. First, note that

$$\mathrm{c}\ell \otimes 1 : \mathbb{C}\ell(M) \otimes E \otimes F \to E \otimes F$$

gives $E \otimes F$ the structure of a Hermitian Clifford module. Then recall that if ∇^F is any connection on F, we define the **tensor product connection** on $E \otimes F$ by

$$\nabla^{E\otimes F}(e \otimes f) = (\nabla^E e) \otimes f + e \otimes \nabla^F f.$$

It is straightforward to verify that if ∇^F is unitary, then $\nabla^{E\otimes F}$ is unitary with respect to the inner product $\langle e_1 \otimes f_1, e_2 \otimes f_2\rangle = \langle e_1, e_2\rangle\langle f_1, f_2\rangle$ and satisfies the property (3.16) of being a Clifford connection.

Definition 3.16. Let E, D and F be as above. The operator

$$D_F = \mathrm{c}\ell \otimes 1 \circ \nabla^{E\otimes F} \in \mathrm{Diff}^1(M; E \otimes F)$$

is a **twisted Dirac operator** on $E\otimes F$ which we say is obtained by "twisting" $D \in \mathrm{Diff}^1(M; E)$ by F.

Since the representation theory of $\mathbb{C}\ell_n$ is so simple, this leads us to the natural question of if and when arbitrary Dirac operators can be expressed as twistings of some fundamental Dirac operator or operators. The answer to this question involves the notion of *spin structures*.

Consider for simplicity the case that M has dimension $2n$. We recall that any representation W of $\mathbb{C}\ell_{2n}$ decomposes as a multiple of the fundamental irreducible representation $\mathrm{S} = \mathrm{S}_{2n}$; explicitly

$$W \cong \mathrm{S} \otimes H, \tag{3.19}$$

where $H = \mathrm{Hom}_{\mathbb{C}\ell_{2n}}(\mathrm{S}, W)$ has the trivial action. The isomorphism is given by $s \otimes h \mapsto h(s)$. Because of this, it is tempting to suppose that every Clifford module $E \to M$ likewise decomposes as a multiple of some fundamental one, say something like "$\mathbb{C}\ell(M) \times_{\mathbb{C}\ell_{2n}} \mathrm{S}_{2n}$".

There are several problems with this. For one, the latter object is not quite well-defined, since $\mathbb{C}\ell_{2n}$ is an algebra, not a group, and $\mathbb{C}\ell(M)$ is not in fact a principal bundle (though this could be remedied by passing to the subgroup of units). More seriously, we recall that $\mathbb{C}\ell(M)$ is fundamentally associated to the principal bundle $P_{\mathrm{SO}} \to M$ (in principal bundle language, the *structure group* of $\mathrm{C}\ell(M)$ has been *reduced* to $\mathrm{SO}(2n)$), so in order to form a bundle associated to S_{2n} with the right properties, we would need a nontrivial action of $\mathrm{SO}(2n)$ on S. In fact there isn't one, and the examination of this failure leads to a topological obstruction to defining a fundamental irreducible $\mathbb{C}\ell(M)$ module.

3.2.3 Spin

Fix a real inner product space V. We begin by trying to find a group inside $\mathrm{C}\ell(V)$ which acts by special orthogonal transformations on V. For $v, w \in V$, the Clifford relation can be written

$$vw = -wv - 2\langle v, w\rangle.$$

If $v \neq 0$ then v is a unit inside $\mathrm{C}\ell(V)$, with $v^{-1} = -v/\left|v\right|^2$. Thus composing with v^{-1} on the right gives the **adjoint action**

$$\mathrm{Ad}_v(w) = vwv^{-1} = -w + 2\frac{\langle v, w\rangle}{\left|v\right|^2} v.$$

In fact, because of the signs it is better to consider the **twisted adjoint action**

$$\widetilde{\mathrm{Ad}}_v(w) = -vwv^{-1} = w - 2\frac{\langle v, w\rangle}{\left|v\right|^2} v, \tag{3.20}$$

which we recognize as the reflection of w across the hyperplane $v^\perp \subset V$. This is an orthogonal transformation of V, and does not depend on the magnitude of v, i.e., $\widetilde{\mathrm{Ad}}_v(w) = \widetilde{\mathrm{Ad}}_{av}(w)$ for all $a \neq 0$. So we may as well restrict attention to those v with $|v| = 1$.

Definition 3.17. The **Pin group**[7] is the group generated by the unit norm elements of V:

$$\mathrm{Pin}(V) = \{v_1 \dots v_N : v_i \in V, |v_i| = 1\} \subset \mathrm{C}\ell(V).$$

Extending $\widetilde{\mathrm{Ad}}$ multiplicatively defines a homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(V) \to \mathrm{O}(V)$. Since the product of an even number of reflections always preserves the orientation of V, we define the **Spin group** by

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{C}\ell^0(V) = \{v_1 \dots v_{2N} : v_i \in V, |v_i| = 1\},$$

and then $\widetilde{\mathrm{Ad}}$ restricts to a homomorphism[8] $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(V) \to \mathrm{SO}(V)$. Note that, since $\mathrm{Spin}(V) \subset \mathrm{C}\ell^0(V)$, the adjoint and twisted adjoint actions coincide: $\mathrm{Ad} = \widetilde{\mathrm{Ad}}\mathrm{Spin}(V) \to \mathrm{SO}(V)$.

Proposition 3.18. *The twisted adjoint homomorphism is surjective, and defines the short exact sequences*

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(V) \to \mathrm{O}(V) \to 1$$
$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1$$

of groups. In fact, $\mathrm{Spin}(V)$ *is the universal cover of* $\mathrm{SO}(V)$ *for* $\dim(V) \geq 3$.

Proof. Recall that every element $A \in \mathrm{O}(V)$ can be represented as a finite product of reflections, with $A \in \mathrm{SO}(V)$ if and only if the number of reflections is even. Indeed, considering $A \in \mathrm{O}(V)$ as a unitary transformation of $V \otimes \mathbb{C}$, we may diagonalize A by the spectral theorem, and since A is real, the eigenvalues come in complex conjugate pairs. Passing back to the real subspaces of the paired eigenspaces exhibits A as a block diagonal matrix

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix}$$

where the A_i are orthogonal transformations of 1 and 2 dimensional subspaces, in which dimensions the claim can be easily verified by hand. This proves surjectivity of $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(V) \to \mathrm{O}(V)$, and clearly $\mathrm{Spin}(V) = \widetilde{\mathrm{Ad}}^{-1}\big(\mathrm{SO}(V)\big)$.

To see that $\ker \widetilde{\mathrm{Ad}} = \{\pm 1\}$, suppose $\widetilde{\mathrm{Ad}}_\eta = 1$. From (3.20), this is equivalent to

$$(-1)^k \eta w = w \eta \quad \forall w \in V, \quad \eta \in \mathrm{Pin}(V) \cap \mathrm{C}\ell^k(V). \tag{3.21}$$

In terms of the orthonormal basis $\{e_I = e_{i_1} \cdots e_{i_l} : I = \{i_1, \dots, i_l\},\ i_1 < \cdots < i_l\}$ of $\mathrm{C}\ell(V)$, we may write

$$\eta = \sum_{|I|=k} a_I e_I, \quad |a_I| = 1.$$

[7] The name is a joke that will make sense in a second.

[8] Get it now?

Applying (3.21) with $w = e_j$, and using

$$e_I e_j = \begin{cases} (-1)^{|I|} e_j e_I, & j \notin I, \\ (-1)^{|I|+1} e_j e_I, & j \in I \end{cases}$$

we obtain that $a_I = 0$ if $j \in I$. Since j was arbitrary, we conclude $\eta = \pm 1$.

It follows that $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ are double covers of $\mathrm{O}(V)$ and $\mathrm{SO}(V)$, respectively. To see that they are nontrivial double covers, it suffices to show that there is a path from 1 to -1. This may be done explicitly, for example if e_i and e_j are orthonormal, then

$$\begin{aligned}(e_i \cos t/2 + e_j \sin t/2)(-e_i \cos t/2 + e_j \sin t/2) &= \cos t + e_i e_j \sin t \\ &= \exp(t e_i e_j), \quad 0 \le t \le \pi,\end{aligned} \tag{3.22}$$

does the trick. (For the second neat equality, notice that $(e_i e_j)^2 = -1$.) Since $\pi_1(\mathrm{SO}(V)) = \mathbb{Z}_2$ if $\dim(V) \ge 3$, it follows that $\mathrm{Spin}(V)$ is the universal cover of $\mathrm{SO}(V)$. $\square$

For later reference the following observation will be useful. Recall that there is a natural isomorphism of vector spaces $\Lambda^2 V \cong \mathfrak{so}(V)$ under which $v \wedge w$ is identified with the skew adjoint transformation

$$(v \wedge w) z = \langle v, z \rangle w - \langle w, z \rangle v.$$

In particular, for $V = \mathbb{R}^n$, $e_i \wedge e_j$ is identified with the standard basis matrix $E_{ij} \in \mathfrak{so}(n)$ having -1 at index (i, j), $+1$ at index (j, i) and 0 everywhere else.

Now, we have a copy of $\Lambda^2 V$ sitting inside $\mathrm{C}\ell(V)$ via (3.4) and the following result identifies this as the Lie algebra of $\mathrm{Spin}(V)$, but with a nontrivial multiplicative factor which is at once potentially confusing and very important!

Proposition 3.19. *The double covering* $\mathrm{Ad} : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ *generates a Lie algebra isomorphism*

$$\mathrm{ad} : \Lambda^2 V = \mathfrak{spin}(V) \xrightarrow{\cong} \mathfrak{so}(V),$$

under which

$$\mathrm{ad}^{-1}(v \wedge w) = \frac{1}{4}[v, w] \in \Lambda^2 V \subset \mathrm{C}\ell(V).$$

In particular, for $V = \mathbb{R}^n$,

$$\mathrm{ad}^{-1}(e_i \wedge e_j) = \frac{1}{2} e_i e_j, \quad i < j, \tag{3.23}$$

(and then $\frac{1}{2} e_i e_j = \frac{1}{4}(e_i e_j - e_j e_i) = \frac{1}{4}[e_i, e_j]$*).*

Remark. Another (slightly more intrinsic) way to understand the factor of $\frac{1}{2}$ is to note that it is necessary to realize a *Lie algebra* isomorphism $\mathfrak{so}(n) \cong \Lambda^2 \mathbb{R}^n \subset \mathrm{C}\ell(\mathbb{R}^n)$. Indeed, we may identify E_{ij} with $e_i \wedge e_j \cong e_i e_j$, but in $\mathfrak{so}(n)$ we have $[E_{ij}, E_{jk}] = E_{ik}$ while in $\mathrm{C}\ell_n$ we have $[e_i e_j, e_j e_k] = 2 e_i e_k$.

Proof. Fix an identification $V \cong \mathbb{R}^n$ and consider the curve

$$\gamma(t) = \exp(te_ie_j) = \cos t + e_ie_j \sin t \in \mathrm{Spin}(n)$$

(see (3.22)) with $\gamma(0) = 1$, $\gamma'(0) = e_ie_j$. This shows that $\Lambda^2 V \subset \mathfrak{spin}(V)$, and since the dimensions of these linear spaces are the same, we must have equality. To compute the image of e_ie_j under $\mathrm{ad} : \mathfrak{spin}(V) \to \mathfrak{so}(V)$, consider $\mathrm{Ad}_{\gamma(t)}(v)$ for some $v \in \mathbb{R}^n$. We have

$$\begin{aligned}
\mathrm{ad}_{e_ie_j} v &= \frac{d}{dt}|_{t=0} \mathrm{Ad}_{\gamma(t)}(v) = \frac{d}{dt}|_{t=0}\gamma(t)v\gamma^{-1}(t) \\
&= e_ie_jv - ve_ie_j = e_ie_jv + (e_iv + 2\langle v, e_i\rangle)e_j \\
&= e_ie_jv - e_ie_jv + 2\langle v, e_i\rangle)e_j - 2\langle v, e_j\rangle e_i = 2(e_i \wedge e_j)v
\end{aligned}$$

which proves (3.23), and then writing $\frac{1}{2}e_ie_j = \frac{1}{4}[e_i, e_j]$ and extending bilinearly gives the general result. □

Remark. This suggests an alternative way to construct $\mathrm{Spin}(V)$. Instead of looking initially for an orthogonal group representation on V, we can look for a Lie algebra representation. One can then show directly that $[\Lambda^2 V, \Lambda^2 V] \subset \Lambda^2 V \subset \mathrm{C}\ell(V)$, so $\Lambda^2 V$ is a Lie subalgebra of $\mathrm{C}\ell(V)$ with bracket the commutator. In addition, $[\Lambda^2 V, \Lambda^1 V] \subset \Lambda^1 V$, so it acts on $V = \Lambda^1 V \subset \mathrm{C}\ell(V)$, and it is easy enough to show that the action is infinitesimally orthogonal. Then we can define $\mathrm{Spin}(V) = \exp(\Lambda^2 V) \subset \mathrm{C}\ell(V)$ by exponentiating inside the Clifford algebra. See [BGV92] for details.

Consider the irreducible graded $\mathbb{C}\ell_{2n}$ representation $\mathrm{S}_{2n} = \mathrm{S}_{2n}^+ \oplus \mathrm{S}_{2n}^-$. Since $\mathrm{Spin}(2n) \subset \mathbb{C}\ell_{2n}^0$, it follows that S_{2n}^+ and S_{2n}^- are representations of $\mathrm{Spin}(2n) := \mathrm{Spin}(\mathbb{R}^{2n})$, which are inequivalent. In fact, since they are irreducible representations of $\mathbb{C}\ell_{2n}^0$ and $\mathrm{Spin}(2n)$ contains a basis for the latter space they must be irreducible representations for $\mathrm{Spin}(2n)$. Likewise in the odd case, the two inequivalent (ungraded) representations S_{2n+1}^+ and S_{2n+1}^- lead to inequivalent irreducible representations of $\mathrm{Spin}(2n + 1)$. We conclude

Corollary 3.20. *The half-spinor representations* S_n^+ *and* S_n^- *are irreducible representations of* $\mathrm{Spin}(n)$ *which do not factor through representations of* $\mathrm{SO}(n)$.

In order to obtain a vector bundle over M associated to S, it is therefore necessary to work with a principal $\mathrm{Spin}(n)$ bundle rather than a principal $\mathrm{SO}(n)$ bundle.

3.2.4 Spin structures on a manifold

Definition 3.21. Let M be an oriented Riemannian manifold of dimension n. A **spin structure** on M is a principal $\mathrm{Spin}(n)$ bundle $P_{\mathrm{Spin}} \to M$ which lifts the frame bundle P_{SO}, i.e., there is a morphism $P_{\mathrm{Spin}} \to P_{\mathrm{SO}}$ such that

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r] \arrow[d] & P_{\mathrm{Spin}} \arrow[d] \\ \mathrm{SO}(n) \arrow[r] \arrow[dr] & P_{\mathrm{SO}} \arrow[d] \\ & M \end{tikzcd}$$

commutes, where the left vertical map is the double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. If M admits a spin structure, then we say M is a **spin manifold**.

In general spin structures need not exist due to a cohomological obstruction, as per the following result which we state without proof.

Proposition 3.22. *A manifold M is spin if and only if the second Steifel-Whitney class $w_2(M) \in H^2(M;\mathbb{Z}_2)$ vanishes. Moreover, if $w_2(M) = 0$, then the spin structures on M are in bijective correspondence with the set $H^1(M;\mathbb{Z}_2)$.*

Remark. For comparison, recall that M is orientable if and only if $w_1(M) = 0 \in H^1(M;\mathbb{Z}_2)$, and then the possible orientations are in bijective correspondence with the set $H^0(M;\mathbb{Z}_2)$. Thus a spin structure can be thought of as a kind of "higher level" orientation.

While we have not covered the general theory of connections on principal bundles, it is a fact that, since $P_{\mathrm{Spin}} \to P_{\mathrm{SO}}$ has discrete fibers (as a double cover), the Levi-Civita connection lifts to a unique connection on P_{Spin}, and hence on any bundle associated to it.

Definition 3.23. If M is a spin manifold of even dimension $2n$, then the **spinor bundle** $\mathbb{S}(M) \to M$ is the graded vector bundle

$$\mathbb{S}(M) = \mathbb{S}^+(M) \oplus \mathbb{S}^-(M) = P_{\mathrm{Spin}} \times_\rho (\mathbb{S}^+_{2n} \oplus \mathbb{S}^-_{2n})$$

where $\rho = \rho_+ \oplus \rho_- : \mathrm{Spin}(n) \to \mathbb{S}_{2n} = \mathbb{S}^+_{2n} \oplus \mathbb{S}^-_{2n}$ is the representation of $\mathrm{Spin}(2n)$ from Corollary 3.20.

It carries a canonical Levi-Civita connection $\nabla^{\mathrm{LC}} : C^\infty\big(M;\mathbb{S}^\pm(M)\big) \to C^\infty\big(M;T^*M \otimes \mathbb{S}^\pm(M)\big)$, which satisfies (3.16) with respect to the Clifford action $\mathrm{c}\ell : \mathbb{C}\ell(M) \otimes \mathbb{S}(M) \to \mathbb{S}(M)$, and the **spin Dirac operator**

$$ð = \begin{pmatrix} 0 & ð^+ \\ ð^- & 0 \end{pmatrix} \in \mathrm{Diff}^1\left(M;\mathbb{S}^+(M) \otimes \mathbb{S}^-(M)\right) \tag{3.24}$$

is then defined by $ð = \mathrm{c}\ell \circ \nabla^{\mathrm{LC}}$, where $\mathrm{c}\ell : \mathbb{C}\ell_n \to \mathrm{End}(\mathbb{S}_{2n})$ is the fundamental Clifford representation.

More generally, we now have a functorial way of constructing Clifford modules over M in any dimension. Indeed, suppose $(W = W^0 \oplus W^1, \mathrm{c}\ell)$ is any graded $\mathbb{C}\ell_n$ representation. The components W^j restrict to representations $\mathrm{c}\ell|_{\mathrm{Spin}(n)}$ of $\mathrm{Spin}(n)$, and we may form the graded bundle

$$E = E^0 \oplus E^1 = P_{\mathrm{Spin}} \times_{\mathrm{c}\ell\oplus\mathrm{c}\ell} (W^0 \oplus W^1) \to M.$$

Note that the adjoint action $\mathrm{Ad} = \widetilde{\mathrm{Ad}}$ of $\mathrm{Spin}(n)$ on $\mathbb{R}^n$ extends to the adjoint action of $\mathrm{Spin}(n)$ on $\mathbb{C}\ell_n = \mathbb{C}\ell(\mathbb{R}^n)$ itself, and we have a commutative diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}} \times \mathbb{C}\ell_n \otimes W & \xrightarrow{\mathrm{c}\ell} & P_{\mathrm{Spin}} \times W \\ \Big\downarrow a & & \Big\downarrow a \\ P_{\mathrm{Spin}} \times \mathbb{C}\ell_n \otimes W & \xrightarrow{\mathrm{c}\ell} & P_{\mathrm{Spin}} \times W \end{array}$$

where the vertical arrows are the action of $a \in \mathrm{Spin}(n)$:

$$a \cdot \big(p, \eta \otimes v\big) = \big(pa^{-1}, \mathrm{Ad}_a\, \eta \otimes \mathrm{c}\ell(a)v\big) = \big(pa^{-1}, a\eta a^{-1} \otimes \mathrm{c}\ell(a)v\big),$$
$$\big(pa^{-1}, \mathrm{c}\ell(a\eta a^{-1})\mathrm{c}\ell(a)v\big) = \big(pa^{-1}, \mathrm{c}\ell(a)\mathrm{c}\ell(\eta)v\big).$$

Thus the Clifford action descends to the quotient and gives E the structure of a graded Clifford module:

$$\mathrm{c}\ell : \mathbb{C}\ell(M) \otimes E \to E.$$

In addition, the lift of the Levi-Civita connection to P_{Spin} determines a connection $\nabla : C^\infty(M; E) \to C^\infty(M; T^*M \otimes E)$ on E which is easily seen to be a Clifford connection.

Note that the Koszul formula (3.15) and Proposition 3.19 give a way to compute the action of ∇ locally on any such Clifford module. Indeed, fixing a local orthonormal frame $\{e_1, \dots, e_n\}$ for TM over an open set $U \subset M$ (which is a local section of P_{SO}), we can write

$$\nabla_{e_i} e_j = \sum_k \Gamma_{kij} e_k, \quad \Gamma_{kij} = \tfrac{1}{2}(\langle [e_i, e_j], e_k\rangle - \langle [e_j, e_k], e_i\rangle + \langle [e_k, e_i], e_j\rangle) \tag{3.25}$$

with the coefficients computed from (3.15). Then the covariant derivatives, as differential operators on $C^\infty(U; \mathbb{R}^n)$ (we have trivialized TM over U), take the form

$$\nabla_{e_i} = e_i + \sum_{j<k} \Gamma_{kij}(e_j \wedge e_k) \in \mathrm{Diff}^1(U; \mathbb{R}^n)$$

where the first term is the vector field e_i considered as a first order differential operator, and the second term is in $C^\infty\big(U; \mathfrak{so}(n)\big)$ where we write an element of $\mathfrak{so}(n)$ as an element of $\Lambda^2\mathbb{R}^n$.

Choosing a lift of the local section from P_{SO} to P_{Spin} gives a local trivialization of E, and then using Proposition 3.19 to lift $e_j \wedge e_k \in \mathfrak{so}(n)$ to $\mathrm{ad}^{-1}(e_j \wedge e_k) = \frac{1}{2} e_j e_k$ of $\mathfrak{spin}(n)$ and applying derivative of the representation $\mathrm{c}\ell : \mathrm{Spin}(n) \to W$, we have the local equation

$$\nabla^E_{e_i} = e_i + \frac{1}{2} \sum_{j<k} \Gamma_{kij} \mathrm{c}\ell(e_j\, e_k) \in C^\infty\big(U; \mathrm{End}(E)\big). \tag{3.26}$$

There are two choices for the lift of the local section of P_{SO} to the double cover P_{Spin}, but they amount to the same local formula. Let us pause to record what we have shown.

Proposition 3.24. *There is a natural association between (graded) $\mathbb{C}\ell_n$ representations and (graded) Clifford modules on a spin manifold M. If $E = P_{\mathrm{Spin}} \times_\rho W$ is the Clifford module associated to a representation (W, ρ), then E admits a canonical (graded) Dirac operator*

$$D = \mathrm{c}\ell \circ \nabla \in \mathrm{Diff}^1(M; E).$$

The Clifford connection on E, induced by the lift of the Levi-Civita connection on P_{Spin}, acts locally on sections by (3.26).

Since any $\mathbb{C}\ell_{2n}$ representation W breaks up into a multiple of the fundamental irreducible representation S_{2n} via (3.19), we have the following result:

Proposition 3.25. *Let M be a spin manifold of even dimension. Then any graded Clifford module $E^0 \oplus E^1 \to M$ has the form*

$$E \cong \mathrm{S}(M) \otimes H = \big(\mathrm{S}^+(M) \oplus \mathrm{S}^-(M)\big) \otimes H, \quad H = \mathrm{Hom}_{\mathbb{C}\ell(M)}\big(\mathrm{S}(M), E\big) \tag{3.27}$$

where H carries the trivial action of $\mathbb{C}\ell(M)$.

Moreover, every graded Dirac operator $D \in \mathrm{Diff}^1(M; E^0 \oplus E^1)$ is a twisting $D = \eth_H$ of the spin Dirac operator with respect to some connection on H.

Proof. The formula (3.27) is immediate: the map in the other direction is via $s \otimes h \mapsto h(s)$ and the fact that is an isomorphism follows from the analogous statement fiberwise. Suppose then that $D = \mathrm{c}\ell_E \circ \nabla^E$ is a Dirac operator. Under the isomorphism (3.27), $\mathrm{c}\ell_E \cong \mathrm{c}\ell_{\mathrm{S}} \otimes 1$, and there is a unique connection ∇^H on H such that $\nabla^E \cong \nabla^{\mathrm{S}} \otimes 1 + 1 \otimes \nabla^H$ holds. $\square$

Remark. Note that, while a spin structure may not exist on M, it always exists locally. Thus, for any Clifford module $E \to M$, if we restrict to a sufficiently small set $U \subset M$, then we can always suppose $E \cong \mathrm{S}(U) \otimes H$ as above. This is useful in several contexts, for instance to show that a Clifford module always admits a Clifford connection. Indeed, taking any connection on H and the Levi-Civita connection on $\mathrm{S}(U)$ defines a Clifford connection on the product, and then these local connections can be summed over M using a partition of unity.

It is also useful in the context of the index theorem. Indeed, we will mostly focus on proving the index theorem for the spin Dirac operator and its twistings, but since the result is local (in the sense that the index is computed as an integral over M of some index *density*), it is actually valid for arbitrary Dirac operators over non-spin manifolds.

3.2.5 Curvature and Bochner formulas

Recall that a connection ∇ on an arbitrary vector bundle $E \to M$ has an associated *curvature*, which may be defined in various ways. For instance, the covariant derivative $\nabla : C^\infty(M; E) \to C^\infty(M; T^*M \otimes E)$ extends to a unique **exterior covariant derivative**

$$\widetilde{\nabla} : C^\infty(M; \Lambda^k \otimes E) \to C^\infty(M; \Lambda^{k+1} \otimes E)$$

defined inductively by the property

$$\widetilde{\nabla}(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \widetilde{\nabla} s, \quad \omega \in C^\infty(M; \Lambda^k M), \quad s \in C^\infty(M; E).$$

Then the **curvature** of ∇ is defined to be the transformation

$$K = \widetilde{\nabla} \circ \nabla : C^\infty(M; E) \to C^\infty(M; \Lambda^2 \otimes E).$$

Alternatively, K can be defined in terms of ∇ directly via

$$K(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} : C^\infty(M; E) \to C^\infty(M; E), \quad X, Y \in C^\infty(M; TM). \tag{3.28}$$

From this latter formula (or otherwise) it can be shown that $K \in \mathrm{Diff}^0(M; E, \Lambda^2 \otimes E)$ is actually an operator of order 0, i.e., it is represented by an $\mathrm{End}(E)$-valued 2-form which acts multiplicatively on $C^\infty(M; \Lambda^k \otimes E)$.

Exercise 3.2. Use the symbolic theory of differential operators to show that K has order 0. Hint: the commutator $[\nabla_X, \nabla_Y]$ of two first order differential operators must have first order since the second order principal symbol vanishes. Then convince yourself that the principal symbols of $[\nabla_X, \nabla_Y]$ and $\nabla_{[X,Y]}$ agree.

In particular, for $E = TM$, $\nabla = \nabla^{\mathrm{LC}}$, the curvature is the **Riemann curvature tensor**

$$R \in C^\infty\big(M; \Lambda^2 M \otimes \mathrm{End}(TM)\big) \cong C^\infty(M; \Lambda^2 M \otimes T^*M \otimes TM).$$

The fact that R is a 2-form which operates on TM makes it notationally rather confusing. We will often write $R_{X,Y}(Z)$ instead of $R(X,Y)(Z)$ to denote the contraction of R as a 2-form with the bivector $X \wedge Y$, acting on a section Z of TM. In terms of a (usually coordinate) frame $\{e_1, \ldots, e_n\}$, the standard convention for the indices of R is (somewhat unfortunately, in my view)

$$R_{lijk} = \big\langle R_{e_j, e_k}(e_i), e_l \big\rangle, \quad \text{or} \quad R^l{}_{ijk} \text{ s.t. } R_{e_j,e_k}(e_i) = \sum_l R^l{}_{ijk} e_l. \tag{3.29}$$

We recall some of the fundamental symmetries of R which will be useful below.

Theorem 3.26. *The Riemann curvature tensor enjoys the following symmetries:*

$$\langle R_{X,Y}(Z), W \rangle = -\langle R_{Y,X}(Z), W \rangle \tag{3.30a}$$

$$\langle R_{X,Y}(Z), W \rangle = -\langle R_{X,Y}(W), Z \rangle \tag{3.30b}$$

$$\langle R_{X,Y}(Z), W \rangle + \langle R_{Y,Z}(X), W \rangle + \langle R_{Z,X}(Y), W \rangle = 0 \tag{3.30c}$$

$$\langle R_{X,Y}(Z), W \rangle = \langle R_{Z,W}(X), Y \rangle \tag{3.30d}$$

Exercise 3.3. Prove Theorem 3.26, (or look it up). Equations (3.30a) and (3.30b) reflect the fact that R is a 2-form valued in skew-adjoint endomorphisms of TM. (3.30c) follows by inserting the torsion identity $[X,Y] = \nabla_X Y - \nabla_Y X$ into the Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$. Then (3.30d) follows from the previous ones.

Taking traces of R leads to simpler curvature tensors which capture partial information about R. The **Ricci curvature tensor** is given by

$$\mathrm{Ric}(X,Y) = \mathrm{tr}\, R_{\cdot,X}(Y)\cdot = \sum_j \big\langle R_{e_j,X}(Y), e_j \big\rangle. \tag{3.31}$$

From (3.30d) it is a symmetric 2-cotensor. The **scalar curvature** is given by

$$\kappa = \mathrm{tr}\,\mathrm{Ric}(\cdot,\cdot) = \sum_k \mathrm{Ric}(e_k,e_k) = \sum_{j,k} \big\langle R_{e_j,e_k}(e_k), e_j \big\rangle = \sum_{j,k} -\big\langle R_{e_j,e_k}(e_j), e_k \big\rangle, \tag{3.32}$$

and is a real-valued function on M.

Just as we lifted the action of ∇^{LC} to bundles associated to P_{Spin} in the paragraphs preceding Proposition 3.24, we may use the same approach to lift the action of the Riemann curvature tensor. Indeed, choosing a local orthonormal frame $\{e_1,\ldots,e_n\}$ for TM over an open set U and using (3.29), we may express $R_{e_i,e_j} \in C^\infty\big(U;\mathfrak{so}(TM)\big)$ as the family of skew adjoint transformations

$$R_{e_i,e_j} = \sum_{k<l} R^l{}_{kij}(e_k \wedge e_l) = \sum_{k<l} \big\langle R_{e_i,e_j}(e_k), e_l \big\rangle (e_k \wedge e_l) \in C^\infty\big(U;\mathfrak{so}(n)\big).$$

Then in light of Proposition 3.19 this lifts to act on a Clifford module $E = P_{\mathrm{Spin}} \times_{\mathrm{c}\ell} W$ by

$$\begin{aligned} R^E_{e_i,e_j} &= \frac{1}{2}\sum_{k<l} R^l{}_{kij}\,\mathrm{c}\ell(e_k e_l) = \frac{1}{2}\sum_{k<l} \big\langle R_{e_i,e_j}(e_k), e_l \big\rangle \mathrm{c}\ell(e_k e_l) \\ &= \frac{1}{4}\sum_{k,l} \big\langle R_{e_i,e_j}(e_k), e_l \big\rangle \mathrm{c}\ell(e_k e_l) \in C^\infty\big(U;\mathrm{End}(E)\big). \end{aligned} \tag{3.33}$$

We know that that if D is a Dirac operator associated to a Clifford connection ∇ on E, then D^2 is a positive Laplace-type operator on E. There is another canonical way of constructing a Laplace-type operator on E out of ∇. Indeed, recall that the principal symbol of ∇ is given by

$$\sigma(\nabla)(x,\xi) : E_x \ni e \mapsto i\xi \otimes e \in T^*_x M \otimes E_x.$$

The adjoint of this operation is

$$\sigma(\nabla)(x,\xi)^* : T^*_x M \otimes E_x \ni \eta \otimes e \mapsto -i\langle \xi, \eta \rangle e \in E_x,$$

and then $\sigma(\nabla^*\nabla)(x,\xi) : e \mapsto |\xi|^2\, e$, so the **connection Laplacian**, or **Bochner Laplacian**

$$\nabla^*\nabla \in \mathrm{Diff}^2(M;E)$$

is also a Laplace-type operator, which is evidently positive. The difference of these two Laplacians is the content of the following result, variously referred to as the **general Bochner formula**[9] or the **Weitzenbock formula**.

Theorem 3.27. *Let ∇ be a Clifford connection on a Hermitian Clifford module $E \to M$ with associated Dirac operator $D \in \mathrm{Diff}^1(M;E)$. Then*

$$D^2 = \nabla^*\nabla + \mathcal{R}, \quad \mathcal{R} \in C^\infty\big(M; \mathrm{End}(E)\big), \tag{3.34}$$

where the **curvature term**, *$\mathcal{R}$, is given locally in terms of an orthonormal frame $\{e_1, \dots, e_n\}$ of TM via*

$$\mathcal{R}(s) = \tfrac{1}{2}\sum_{j,k} \mathrm{c}\ell(e_je_k)K(e_j,e_k)(s) = \sum_{j<k} \mathrm{c}\ell(e_je_k)K(e_j,e_k)(s), \quad s \in C^\infty(M;E), \tag{3.35}$$

where K is the curvature tensor of the connection ∇.

Proof. First, let us determine a local formula for $\nabla^*\nabla$. In terms of an orthonormal frame $\{e_1, \dots, e_n\}$, we have

$$\langle \nabla^*\nabla s, t\rangle = \langle \nabla s, \nabla t\rangle = \sum_j \big\langle \nabla_{e_j}s, \nabla_{e_j}t\big\rangle = \sum_j -\big\langle \nabla_{e_j}\nabla_{e_j}s, t\big\rangle + \big\langle \mathrm{div}(e_j)\nabla_{e_j}s, t\big\rangle.$$

We may always choose the frame so that $\mathrm{div}(e_j) = 0$ for all j at a given point x, so it follows that

$$\nabla^*\nabla s = -\sum_j \nabla_{e_j}\nabla_{e_j}s$$

since the right hand side may be expressed as $-\operatorname{tr}\widetilde{\nabla}.\nabla.s$, which is independent of the choice of basis.

Now, taking an orthonormal frame in which $\nabla_{e_j}e_k = 0$ and $[e_j, e_k] = 0$ for all j, k at $x \in M$, we compute

$$\begin{aligned}
D^2 s &= \sum_{j,k} \mathrm{c}\ell(e_j)\nabla_{e_j}\mathrm{c}\ell(e_k)\nabla_{e_k}s \\
&= \sum_{j,k} \mathrm{c}\ell(e_j)\mathrm{c}\ell(e_k)\nabla_{e_j}\nabla_{e_k}s \\
&= \sum_j -\nabla_{e_j}\nabla_{e_j}s + \sum_{j<k}\mathrm{c}\ell(e_je_k)(\nabla_{e_j}\nabla_{e_k} - \nabla_{e_k}\nabla_{e_j})s \\
&= \sum_j -\nabla_{e_j}\nabla_{e_j}s + \sum_{j<k}\mathrm{c}\ell(e_je_k)(\nabla_{e_j}\nabla_{e_k} - \nabla_{e_k}\nabla_{e_j} - \nabla_{[e_j,e_k]})s \\
&= \nabla^*\nabla s + \mathcal{R}s,
\end{aligned}$$

which proves the claim. □

[9]The idea of comparing the Laplacian to the connection Laplacian seems to have originated with Bochner.

There are a few important special cases. First, for the Hodge de Rham operator $D = d + d^* \in \mathrm{Diff}^1(M; \Lambda M)$, we recall that $\Delta = D^2$ actually preserves form degree, so we have $D^2 \in \mathrm{Diff}^1(M; \Lambda^k)$ for each fixed k. Clearly $\nabla^*\nabla$ also preserves form degree, and so too therefore must the curvature term $\mathcal{R}$.

Proposition 3.28. *On 1-forms, the Bochner formula has the form*

$$\Delta = \nabla^*\nabla + \mathrm{Ric} \in \mathrm{Diff}^1(M; \Lambda^1) \tag{3.36}$$

where the **Ricci curvature transformation**, Ric*, is the (dual of the) transformation associated to the bilinear form defined by the Ricci curvature:*

$$\mathrm{Ric}(\phi) := \mathrm{Ric}(\phi^\sharp, \cdot) = \sum_j \big\langle R_{e_j,\phi^\sharp}(\cdot), e_j \big\rangle = -\sum_j \big\langle R_{e_j,\phi^\sharp}(e_j), \cdot \big\rangle \in C^\infty(M; T^*M) \cong C^\infty(M; TM).$$

(Here $\phi^\sharp \in C^\infty(M; TM)$ is obtained from $\phi \in C^\infty(M; T^*M)$ by the metric isomorphism $g : T^*M \cong TM$.)

Proof. We may regard $\phi \in C^\infty(M; \Lambda^1)$ as a section of $\mathrm{C}\ell(M)$, on which the Clifford action is just given by left multiplication. Then the curvature term is given by

$$\mathcal{R}(\phi) = \tfrac{1}{2} \sum_{i,j} e_i e_j R_{e_i,e_j}(\phi) = \tfrac{1}{2} \sum_{i,j,k} e_i e_j e_k \big\langle R_{e_i,e_j}(\phi), e_k \big\rangle.$$

Notice that, when i, j, and k are distinct, $e_i e_j e_k \in \Lambda^3 M \subset \mathrm{C}\ell(M)$, while as remarked above, we know that $\mathcal{R}$ must preserve form degree, so that $\mathcal{R}(\phi) \in \Lambda^1 M \subset \mathrm{C}\ell(M)$. Thus the terms in the sum with i, j and k all distinct must vanish identically[10]. By antisymmetry i and j must be distinct, so we have

$$\begin{aligned}
\mathcal{R}(\phi) &= \tfrac{1}{2} \sum_{i,j} e_i e_j e_i \big\langle R_{e_i,e_j}(\phi), e_i \big\rangle + \tfrac{1}{2} \sum_{i,j} e_i e_j e_j \big\langle R_{e_i,e_j}(\phi), e_j \big\rangle \\
&= \tfrac{1}{2} \sum_{i,j} + e_j \big\langle R_{e_i,e_j}(\phi), e_i \big\rangle + \tfrac{1}{2} \sum_{i,j} - e_i \big\langle R_{e_i,e_j}(\phi), e_j \big\rangle \\
&= \sum_{i,j} - e_i \big\langle R_{e_i,e_j}(\phi), e_j \big\rangle = \sum_{i,j} - e_i \big\langle R_{\phi,e_j}(e_i), e_j \big\rangle \\
&= \sum_{i,j} - e_i \big\langle R_{e_j,\phi}(e_j), e_i \big\rangle = \sum_j - R_{e_j,\phi}(e_j) = \mathrm{Ric}(\phi).
\end{aligned}$$

∎

As a Corollary, we deduce a topological obstruction to a manifold having positive Ricci curvature, a result originally due to Bochner.

[10] Alternatively, you can swap ϕ and e_k in these terms and use the Bianchi identity (3.30c) to show that these terms vanish. Or, you can regard this argument as a proof of (3.30c)!

Corollary 3.29 (Bochner). *Let M be a compact Riemannian manifold with* $\mathrm{Ric} \geq 0$ *(pointwise as a bilinear form on TM). If* $\mathrm{Ric} > 0$ *at some point $x \in M$, then $H^1(M;\mathbb{R}) = 0$.*

Proof. Suppose that $H^1(M;\mathbb{R}) = 0$. By Hodge theory, this means there is a nontrivial 1-form $\phi \in C^\infty(M;\Lambda^1)$ which is harmonic: $\Delta\phi = 0$. Pairing $\Delta\phi$ with ϕ, using (3.36) and integrating, we obtain

$$0 = (\Delta\phi, \phi) = \|\nabla\phi\|^2 + \int_M \mathrm{Ric}(\phi, \phi).$$

In particular, $\int_M \mathrm{Ric}(\phi,\phi) \leq 0$. Since $\mathrm{Ric} \geq 0$ by hypothesis, it must be that $\mathrm{Ric} = 0$ identically, but this contradicts the hypothesis that $\mathrm{Ric} > 0$ at some point. ☐

The second case that we want to consider is the spin Dirac operator, in which case the result is known as the **Lichnerowicz formula**.

Proposition 3.30 (Lichnerowicz). *On spinors, the spin Laplacian and connection Laplacian are related by*

$$\eth^2 = \nabla^*\nabla + \tfrac{1}{4}\kappa,$$

where κ denotes the scalar curvature.

Proof. For fixed e_i, e_j, the local curvature endomorphism R_{e_i,e_j} acts on $\mathbb{S}(M)$ by (3.33). For notational simplicity, we write $c\ell(e_i)$ simply as left multiplication by e_i. Then

$$\mathcal{R}^{\mathbb{S}} = \frac{1}{2}\sum_{i,j} e_i e_j R_{e_i,e_j} = \frac{1}{8}\sum_{i,j,k,l} e_i e_j \langle R_{e_i,e_j} e_k, e_l\rangle e_k e_l = \frac{1}{8}\sum_{i,j,k,l} \langle R_{e_i,e_j} e_k, e_l\rangle e_i e_j e_k e_l. \tag{3.37}$$

We may split off from the sum the terms where i, j, and k are all distinct, and then use

$$\sum_l \sum_{\substack{i,j,k\\ \text{distinct}}} \langle R_{e_i,e_j} e_k, e_l\rangle e_i e_j e_k e_l$$
$$= \sum_l \frac{1}{3}\Big(\sum_{\substack{i,j,k\\ \text{distinct}}} \langle R_{e_i,e_j} e_k, e_l\rangle e_i e_j e_k + \langle R_{e_j,e_k} e_i, e_l\rangle e_j e_k e_i + \langle R_{e_k,e_i} e_j, e_l\rangle e_k e_i e_j \Big) e_l = 0,$$

which follows from (3.30c). Thus the only terms to consider are where i, j, and k are not all distinct. By antisymmetry of R we must have $i \neq j$, so either $i = k$ or $j = k$. Thus (3.37) becomes

$$\begin{aligned}
\mathcal{R}^{\mathbb{S}} &= \frac{1}{8}\sum_{i,j,l} \big(\langle R_{e_i,e_j}(e_i), e_l\rangle e_i e_j e_i e_l + \langle R_{e_i,e_j}(e_j), e_l\rangle e_i e_j e_j e_l\big) \\
&= \frac{1}{8}\sum_{i,j,l} \big(\langle R_{e_i,e_j}(e_i), e_l\rangle e_j e_l - \langle R_{e_i,e_j}(e_j), e_l\rangle e_i e_l\big) \\
&= \frac{1}{4}\sum_{i,j,l} \langle R_{e_i,e_j}(e_i), e_l\rangle e_j e_l.
\end{aligned}$$

By (3.30d), $\langle R_{e_i,e_j}(e_i), e_l\rangle$ is symmetric with respect to interchanging j and l, while $e_j e_l$ is antisymmetric if $j \neq l$, so we must have $j = l$ and then by (3.32),

$$\mathcal{R}^{\mathrm{S}} = -\frac{1}{4}\sum_{i,j}\langle R_{e_i,e_j}(e_i), e_j\rangle = \frac{1}{4}\kappa. \qquad \square$$

Recall that for a tensor product connection $\nabla^{E\otimes F} = \nabla^E \otimes 1 + 1 \otimes \nabla^F$, the curvature is given by

$$K^{E\otimes F} = K^E \otimes 1 + 1 \otimes K^F.$$

In the case of a twisting $D = \eth_F$ of the spin Dirac operator by a bundle (F, ∇^F) this leads to the following.

Corollary 3.31. *The Bochner formula for a twisted Dirac operator $\eth_F \in \mathrm{Diff}^1(M; \mathrm{S}(M)\otimes F)$ is given by*

$$\eth_F^2 = \nabla^*\nabla + \tfrac{1}{4}\kappa + \mathcal{K}, \quad \mathcal{K} \in C^\infty\big(M; \mathrm{End}(\mathrm{S}(M)\otimes F)\big),$$
$$\mathcal{K}(s\otimes f) = \tfrac{1}{2}\sum_{i,j}\big(\mathrm{c}\ell(e_ie_j)s\big)\otimes\big(K^F(e_i,e_j)f\big).$$

3.2.6 Supertrace

Before we leave the subject of Clifford algebras and in preparation for the heat supertrace computation in the next section, let us say a bit more about general $\mathbb{Z}_2$ graded formalism, and the supertrace in particular. As noted above, a $\mathbb{Z}_2$ graded vector space is simply one of the form $E = E^0 \oplus E^1$. An algebra $\mathcal{A}$ is $\mathbb{Z}_2$ graded if it is graded as a vector space, so $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$, and $\mathcal{A}^i \cdot \mathcal{A}^j \subset \mathcal{A}^{i+j(\mathrm{mod}\ 2)}$.

In particular, the algebra $\mathrm{End}(E) = \mathrm{End}(E^0 \oplus E^1)$ of endomorphisms on a graded vector space is a graded algebra:

$$\mathrm{End}(E) = \mathrm{End}^0(E) \oplus \mathrm{End}^1(E),$$
$$\mathrm{End}^0(E) = \mathrm{Hom}(E^0,E^0)\oplus\mathrm{Hom}(E^1,E^1) \quad \mathrm{End}^1(E) = \mathrm{Hom}(E^0,E^1)\oplus\mathrm{Hom}(E^1,E^0).$$

In other words, the even elements of $\mathrm{End}(E)$ are the block diagonal endomorphisms and the odd elements are the block antidiagonal endomorphisms, and a general endomorphism $A \in \mathrm{End}(E)$ can then be decomposed uniquely as

$$\begin{pmatrix} A_{00} & A_{10} \\ A_{01} & A_{00}\end{pmatrix} = A = A^0 \oplus A^1 \quad A^0 = \begin{pmatrix} A_{00} & 0 \\ 0 & A_{11}\end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & A_{10} \\ A_{01} & 0\end{pmatrix}.$$

The action of $\mathrm{End}(E)$ on E is graded in the sense that $\mathrm{End}^j(E)\cdot E^k \subset E^{j+k(\mathrm{mod}\ 2)}$, and in general, a graded representation of an algebra $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ on $E = E^0 \oplus E^1$ is a homomorphism $\mathcal{A} \to \mathrm{End}(E)$ such that $\mathcal{A}^j \to \mathrm{End}^j(E)$. We have discussed these above in the case that $\mathcal{A} = \mathbb{C}\ell(V)$.

In a graded algebra, a natural replacement for the commutator is the **supercommutator**, defined on elements of pure degree by

$$[A^j, B^k]_s = A^j B^k - (-1)^{jk} B^k A^j, \quad i, j \in \mathbb{Z}_2, \tag{3.38}$$

and then extended by linearity. This is the ordinary commutator except on odd elements, where it becomes the anticommutator. The general rule of thumb in the graded formalism is that, anytime you move "something" past "something else" (i.e., in working with products, derivations and so on), you insert the sign -1 exponentiated by the product of their degrees.

In finite dimensions, we can compute the trace of an endomorphism as usual, but in order to retain the analogue of the key property $\operatorname{tr}([A, B]) = 0$, the trace should be replaced by the **supertrace**

$$\operatorname{str} A := \operatorname{tr} A_{00} - \operatorname{tr} A_{11}, \quad A \in \operatorname{End}(E^0 \oplus E^1).$$

Exercise 3.4. Show that, in finite dimensions, str is the unique linear functional $\operatorname{str} : \operatorname{End}(E^0 \oplus E^1) \to \mathbb{C}$ with the property that it vanishes on supercommutators: $\operatorname{str}([A, B]_s) = 0$.

The relevance for us at the moment is the following situation. Recall that $\mathrm{S}_{2n} = \mathrm{S}^+_{2n} \oplus \mathrm{S}^-_{2n}$ is the unique irreducible representation of $\mathbb{C}\ell_{2n}$, which may be graded in one of two ways. By convention we take $\mathrm{S}^0 = \mathrm{S}^+$ and $\mathrm{S}^1 = \mathrm{S}^-$, where $\mathrm{S}^\pm$ are distinguished as the eigenspaces of the Clifford volume element (3.10):

$$\mathrm{S}^\pm_{2n} \ni s \iff c\ell(\omega_{2n})s = \pm 1, \quad \omega_{2n} = i^n e_1 \cdots e_{2n}.$$

Recall that we deduced the existence of S_{2n} via the isomorphism $\mathbb{C}\ell_{2n} \cong \operatorname{Mat}_{2^n}(\mathbb{C})$, under which $\mathrm{S}_{2n} \cong \mathbb{C}^{2^n}$. In particular, we have the natural identification

$$\operatorname{End}(\mathrm{S}^+_{2n} \oplus \mathrm{S}^-_{2n}) \cong \mathbb{C}\ell_{2n} \tag{3.39}$$

of graded algebras. This identification is fundamental to Getzler's rescaling argument in the proof of the index theorem. In particular, (3.39) permits us to define a natural supertrace on $\mathbb{C}\ell_{2n}$. Recall that $\mathbb{C}\ell_{2n}$ is filtered as an algebra by

$$\mathbb{C}\ell_{2n} = \mathbb{C}\ell^{(2n)} \supset \mathbb{C}\ell^{(2n-1)} \supset \cdots \supset \mathbb{C}\ell^{(0)} = \mathbb{C}.$$

Proposition 3.32. *Under the identification* (3.39)*, the supertrace vanishes on* $\mathbb{C}\ell^{(j)}_{2n}$ *for all* $j < 2n$*. In particular,* str *is only nontrivial on the subspace* $\mathbb{C}\langle e_1 \cdots e_{2n}\rangle \subset \mathbb{C}\ell_{2n}$*, where it is uniquely determined by*

$$\operatorname{str}(\omega_{2n}) = 2^n, \quad \omega_{2n} = i^n e_1 \cdots e_{2n}.$$

Proof. Write $a \in \mathbb{C}\ell$ and denote by $A \in \operatorname{End}(\mathrm{S})$ the image of a under the identification (3.39), so $\operatorname{str}(a) = \operatorname{tr} A_{00} - \operatorname{tr} A_{11}$. If $a \in \mathbb{C}\ell^1$ then $A \in \operatorname{End}^1(\mathrm{S})$ is antidiagonal and $\operatorname{str} a = 0$, so it suffices to consider $a \in \mathbb{C}\ell^0$.

Let $a = e_{i_1} \cdots e_{i_{2l}}$ be a basis element of nonmaximal degree, so $l < n$. Then there exists some e_j such that $j \notin \{i_1, \ldots, i_{2l}\}$. Since $e_j \in \mathbb{C}\ell^1$, the corresponding endomorphism E_j is an isomorphism

$$E_j : \mathbb{S}^{\pm} \xrightarrow{\cong} \mathbb{S}^{\mp}$$

with $E_j^{-1} = -E_j$ in light of $e_j^2 = -1$. Using the Clifford relations, we compute

$$a = e_{i_1} \cdots e_{i_{2n}} = -e_j^2 e_{i_1} \cdots e_{i_{2n}} = -e_j e_{i_1} \cdots e_{i_{2n}} e_j = e_j (e_{i_1} \cdots e_{i_{2n}}) e_j^{-1},$$

and then it follows that we may write

$$A = \begin{pmatrix} A_{00} & 0 \\ 0 & E_j A_{00} E_j^{-1} \end{pmatrix}, \implies \operatorname{str} a = \operatorname{tr} A_{00} - \operatorname{tr}(E_j A_{00} E_j^{-1}) = 0.$$

On $a = \omega_{2n} = i^n e_1 \cdots e_{2n}$, it follows from the definition of $\mathbb{S}^{\pm}_{2n}$ and the fact that $\dim(\mathbb{S}^{\pm}_{2n}) = 2^{n-1}$ that

$$A = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \implies \operatorname{str} \omega_{2n} = 2^n. \qquad \square$$

3.3 Heat kernels and Getzler rescaling

With the conventions for Dirac operators in hand, let us now restate the index problem. Let

$$D = \begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix} \in \operatorname{Diff}^1(M; E)$$

be a graded Dirac operator acting on sections of a graded Clifford module $E = E^0 \oplus E^1$. Thus $D_0 \in \operatorname{Diff}^1(M; E_0, E_1)$ is a Dirac operator with adjoint $D_0^* = D_1 \in \operatorname{Diff}^1(M; E_1, E_0)$, and we wish to compute $\operatorname{ind}(D_0) = \dim \operatorname{Null} D_0 - \dim \operatorname{Null} D_1$. By the McKean-Singer formula observed in §3.1, this is given by

$$\operatorname{ind}(D_0) = \operatorname{Tr} e^{-tD_1 D_0} - \operatorname{Tr} e^{-tD_0 D_1} = \operatorname{Str} e^{-tD^2} \quad \forall t \in \mathbb{R}_+,$$

where the **supertrace** of a trace-class operator $K \in \mathcal{B}_1\big(L^2(M; E)\big)$ is defined with respect to the grading $L^2(M; E) = L^2(M; E^0) \oplus L^2(M; E^1)$ by

$$\operatorname{Str} K = \operatorname{Str} \begin{pmatrix} K_{00} & K_{10} \\ K_{01} & K_{11} \end{pmatrix} = \operatorname{Tr} K_{00} - \operatorname{Tr} K_{11}.$$

This can be written in terms of the ordinary trace by composing with an involution:

$$\operatorname{Str} K = \operatorname{Tr} RK, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and then it follows from Lidskii's Theorem 2.20 that if $K \in C^\infty\big(M^2; \mathrm{END}(E)\big)$ is a smoothing operator[11],

$$\operatorname{Str} K = \int_M \operatorname{tr} RK(x,x)\, \mathrm{dVol}_g = \int_M \operatorname{str} K(x,x)\, \mathrm{dVol}_g,$$

where $\operatorname{str} : C^\infty\big(M; \mathrm{End}(E)\big) \to C^\infty(M)$ is the fiberwise supertrace with respect to the grading $E = E^0 \oplus E^1$.

We constructed the heat kernel $H(t,x,y) = e^{-tD^2}$ for an arbitrary Laplace-type operator in Chapter 2, and in principle we could compute the relevant asymptotic as $t \to 0$ of $\operatorname{Str} e^{-tD^2} = \int_{M_{\mathrm{diag}}} \operatorname{str} H(t,x,x)\, \mathrm{dVol}_g$ from the asymptotic expansion of H at the boundary face $\mathsf{hf} \subset M^2_H$ in the heat space. Unfortunately, as we noted in §3.1, constancy of $\operatorname{Str} e^{-tD^2}$ means that the first nonvanishing term have order $\mathcal{O}(t^0)$, which involves the little supertrace of the nth term in the asymptotic expansion of H at hf, where $n = \dim(M)$. It is all but impossible in practice to explicitly compute a term so far down in the expansion[12], so we must revisit our construction in some way.

The clever idea due to Getzler [BGV92], reformulated by Melrose [Mel93], and simplified by the author in the current presentation, is to *rescale* the bundle $\mathrm{END}(E)$ on M^2_H in such a way that the leading term in the heat kernel, considered as a section of the rescaled bundle, carries the desired information, i.e., has nontrivial supertrace.

3.3.1 Rescaling a bundle at a hypersurface

Here we consider an abstract setup in order to fix ideas. Let X be an oriented manifold, let $F \to X$ be a vector bundle, and denote by $\mathcal{F} = C^\infty(X; F)$ the space of smooth sections[13] of F. This is a module over the algebra $C^\infty(X)$ of smooth functions. The important point here is that we can recover F from $\mathcal{F}$ in the following manner. Let $p \in X$, and denote by

$$\mathcal{I}_p = \{u : u(p) = 0\} \subset C^\infty(X)$$

the set of smooth functions on X which vanish at p; this is an ideal (in fact a maximal ideal) in the algebra $C^\infty(X)$.

Proposition 3.33. *There is a natural isomorphism*

$$F_p \cong \mathcal{F}/\mathcal{I}_p\mathcal{F}$$

exhibiting the fiber space of F at p as the quotient of sections of F by those vanishing at p. Moreover, the C^∞ structure and local triviality of $F = \bigsqcup_{p \in X} F_p$ are induced by $\mathcal{F}$, so $F \to X$ is recovered as a smooth vector bundle from the algebra $\mathcal{F}$.

[11] Or more generally, a trace-class operator whose Schwartz kernel admits a well-defined restriction to the diagonal.

[12] The first heat kernel proofs of the index theorem did this by determining the universal properties such a term must have, and then explicitly computing the index of Dirac operators on sufficiently many example spaces to determine it completely.

[13] Technically speaking, $\mathcal{F}$ is better considered as a **sheaf** on X, where $\mathcal{F}(U) = C^\infty(U; F)$ for an open set $U \subset X$, though since all sheaves we consider here are *flabby*, it will suffice to consider global sections. If you don't know what any of this means, don't worry!

Remark. This is really a somewhat dumbed-down instance of the *Serre-Swan Theorem*, which states that there are equivalences of categories between the category of vector bundles on X, the category of finitely generated projective modules (such as $\mathcal{F}$) over $C^\infty(X)$, and the category of locally free sheaves of $\mathcal{O}_X$ modules, where $\mathcal{O}_X$ denotes the sheaf of smooth functions.

Proof. Let $f \in \mathcal{F}$ be a section of F, and let $x = (x_1, \ldots, x_n)$ denote local coordinates centered at p. Then f has the two term Taylor expansion $f(x) = f(0) + \sum_{i=1}^n x_i f_i(x)$, and since each $x_i = 0$ at p, the image of f in the quotient $\mathcal{F}/\mathcal{I}_p\mathcal{F}$ is just $f(0)$; in particular we have a linear map $\mathcal{F}/\mathcal{I}_p\mathcal{F} \to F_p$ taking f to $f(p)$. Clearly two sections f and g have the same image in the quotient if and only if $f(p) = g(p)$, so the map is well-defined and injective, and it is surjective since for any $v \in F_p$ we can construct a local section having $f(p) = v$ and then extend this to a global section using a smooth cutoff function.

To recover F as a vector bundle from $\mathcal{F}$, note that the above isomorphism allows us to interpret $f \in \mathcal{F}$ as a map $f : X \to F := \bigsqcup_{p\in X} F_p$. The local triviality of F comes from the fact that, for a sufficiently small open set $U \subset X$, there exist $\{f_1, \ldots, f_k\} \in \mathcal{F}$ (a local frame for $F|_U$, extended by 0 on X using smooth cutoff functions) such that any $f \in \mathcal{F}$ with support in U can be written uniquely as $f = \sum_{i=1}^k a_k f_k$. The topology and smooth structure on F are fixed by requiring $C^\infty(X; F) = \mathcal{F}$. $\square$

Now let $Y \subset X$ be an oriented hypersurface. By restricting consideration to one side of Y or the other, we may as well assume Y is a boundary hypersurface of X, a manifold with boundary (or possibly corners). Suppose that

$$F^{(0)} \subset F^{(1)} \subset \cdots \subset F^{(m)} = F \tag{3.40}$$

is a filtration of F, which is defined in a neighborhood[14] of Y.

Fix a normal function x to Y, meaning that x vanishes nondegenerately at Y and nowhere else, and consider the subalgebra of sections

$$\mathcal{G} = \left\{ f \in C^\infty(X; F) : f \in \textstyle\sum_{j=0}^m x^j C^\infty(X; F^{(j)}) \right\} \subset \mathcal{F}.$$

Thus $f \in \mathcal{G}$ if its Taylor expansion normal to Y has the local form

$$f(x, y) = f_0(y) + x f_1(y) + \cdots + x^{m-1} f_{m-1}(y) + x^m f_m(x, y),$$

where the coefficients $f_j(y)$ are sections of $F^{(j)}$ over Y.

Remark. It is not too hard to show (though we do not do so here) that $\mathcal{G}$ is independent of the choice of x. For this it is necessary that (3.40) is a filtration as opposed to a grading, and that the filtration is not defined solely over Y but comes with some adequate notion of extension in the normal direction.

[14] A weaker condition suffices, namely that the filtration is defined up to some finite order jet of Y in X, though we shall not need to use this; see [Mel93]. In the case $m = 1$, it is actually sufficient that the filtration (which is really just a subbundle in this case) be defined just at Y itself.

The claim is that there is a well-defined vector bundle ${}^{\mathrm{R}}F \to X$ such that $\mathcal{G} = C^\infty(X; {}^{\mathrm{R}}F)$ is the space of *all* smooth sections of ${}^{\mathrm{R}}F$, with no restrictions over Y. Indeed, note that $\mathcal{G}$ is again a $C^\infty(X)$ module, and we may define

$$ {}^{\mathrm{R}}F = \bigsqcup_{p \in X} {}^{\mathrm{R}}F_p \to X, \quad {}^{\mathrm{R}}F_p = \mathcal{G}/\mathcal{I}_p\mathcal{G}, $$

Proposition 3.34. *The set ${}^{\mathrm{R}}F$ defined above has a natural structure of a vector bundle over X of rank equal to that of F, and there is a natural map ${}^{\mathrm{R}}F \to F$ of vector bundles, which restricts to an isomorphism over $X \setminus Y$.*

Proof. If $p \in X \setminus Y$ then the inclusion $\mathcal{G} \subset \mathcal{F}$ descends to an isomorphism $\mathcal{G}/\mathcal{I}_p\mathcal{G} \cong \mathcal{F}/\mathcal{I}_p\mathcal{F}$, so it suffices to see what happens for $p \in Y$. Denote by r_l the rank of each subbundle $F^{(l)}$, and choose a local frame $\{f_1, \ldots, f_k\}$ for F near p with the property that for each l,

$$ \{f_1, \ldots, f_{r_l}\} \text{ is a local frame for } F^{(l)} \text{ near } p. \tag{3.41} $$

Then $f \in \mathcal{G} \subset \mathcal{F}$ if and only if it has the local form

$$ f = \sum_{l=0}^{m} \sum_{j=r_{l-1}+1}^{r_l} x^l a_j f_j \quad a_j \in C^\infty(X), \tag{3.42} $$

(set $r_{-1} = 0$) and an element is in $\mathcal{I}_p\mathcal{G}$ if and only if it has the above form where in addition $a_j(p) = 0$ for all j. Thus the image of f in $\mathcal{G}/\mathcal{I}_p\mathcal{G}$ is the k-dimensional vector

$$ [f] \cong \big(a_1(p)f_1(p), \ldots, a_k(p)f_k(p)\big). $$

Said another way, $\{f_1', \ldots, f_k'\}$ determine a formal local frame for ${}^{\mathrm{R}}F$ where $f_j' = x^l f_j$, $r_{l-1} < j \leq r_l$; note that $x^l f_j$ are *nonvanishing* at Y, considered as sections of ${}^{\mathrm{R}}F$. As above, the topology and smooth structure on ${}^{\mathrm{R}}F \to X$ are determined by the requirement that $\mathcal{G} = C^\infty(X; {}^{\mathrm{R}}F)$. You can check that this is independent of the choice of frame among those that have the property (3.41).

The inclusion $\mathcal{G} \subset \mathcal{F}$ induces the natural map ${}^{\mathrm{R}}F \to F$, which just amounts to taking local bases $\{f_1', \ldots, f_k'\}$ for ${}^{\mathrm{R}}F$ as above and remembering that $f_j' = x^l f_j$, regarded now as the product of the function $x^l \in C^\infty(X)$ and the section f_j of F. Thus the map is an isomorphism over $X \setminus Y$, but not over Y. Indeed, locally at $p \in Y$,

$$ {}^{\mathrm{R}}F_p \ni (f_1', \ldots, f_k') \mapsto (f_1, \ldots, f_{r_0}, 0, \ldots, 0) \in F_p, $$

so the map is neither injective nor surjective. □

Definition 3.35. The vector bundle ${}^{\mathrm{R}}F \to X$ is the **rescaling of F at Y** associated to the filtration (3.40).

The bundle ${}^{\mathrm{R}}F$ is "rescaled" from F in the sense that some sections of F which originally vanished over Y (such as the $x^l f_j$) are identified with sections of ${}^{\mathrm{R}}F$ which do not vanish over Y. The point is a somewhat subtle one, but operationally speaking the process is simple: we take local expressions (3.42) and either read them as sections of F, with coefficients $x^l a_j$ times the formal basis vectors f_j, or as sections of ${}^{\mathrm{R}}F$ with coefficients a_j times the formal basis vectors $x^l f_j$.

Recall that anytime we have a filtered algebra/vector space/bundle, there is an **associated graded** algebra/vector space/bundle, defined by

$$\operatorname{Gr} F = \bigoplus_{j=0}^{m} F^{(j)}/F^{(j-1)}$$

(set $F_{-1} = \{0\}$) and there is a natural map $F \to \operatorname{Gr} F$, which is an isomorphism of linear spaces, (usually not of algebras) taking $f \in F^{(j)}$ to $[f]$ in the summand $F^{(j)}/F^{(j-1)}$, with j taken as small as possible.

Example 3.36. We have already seen an example of this in (3.4), where ΛV is the associated graded algebra of $\mathrm{C}\ell(V)$.

Another rather sophisticated example we have seen is the filtered algebra $\operatorname{Diff}(M)$ of differential operators on M, whose associated graded algebra is the commutative algebra $C^\infty\big(M; \operatorname{Sym}(TM)\big) = \bigoplus_{k\in\mathbb{N}} P^k(T^*M)$ of polynomial symbols on T^*M, or equivalently sections of the symmetric tensor bundle of TM.

Proposition 3.37. *With respect to a choice of normal function x, the restriction ${}^{\mathrm{R}}F|_Y$ of the rescaled vector bundle over the hypersurface Y is isomorphic to the associated graded bundle:*

$$\begin{gathered} {}^{\mathrm{R}}F|_Y \xrightarrow{\cong} \operatorname{Gr} F|_Y \\ f = \sum_{l=0}^{m} x^l f_l \mapsto ([f_0], [f_1], \dots, [f_m]). \end{gathered} \tag{3.43}$$

Remark. The isomorphism only depends on $dx \in C^\infty(Y; N^*Y)$, in that if $x' = a\, x$ is another normal function with $dx = dx \in C^\infty(Y; N^*Y)$, i.e., $a|_Y \equiv 1$, then the two maps (3.43) will coincide.

Proof. The restriction of ${}^{\mathrm{R}}F$ to Y can be identified with the vector bundle whose sections over Y are given by the quotient algebra $\mathcal{G}/x\mathcal{G}$. The image of $f = \sum_{l=0}^m x^l f_l$, $f_l \in C^\infty(X; F^{(l)})$ in this quotient is easily seen to be the right hand side of (3.43). □

3.3.2 Getzler rescaling

We turn now to the case of interest. Suppose M is a spin manifold of dimension $2n$, and consider the Dirac operator $\eth \in \operatorname{Diff}^1\big(M; \mathrm{S}\big)$ on spinors. We write S instead of $\mathrm{S}(M)$ in this section for notational clarity. Recall that we have a canonical isomorphism

$$\operatorname{End}(\mathrm{S}) \cong \mathbb{C}\ell(M)$$

under which $\mathrm{End}(\mathrm{S})$ is identified with the Clifford bundle, which carries the filtration

$$\mathbb{C}\ell^{(0)}(M) \subset \mathbb{C}\ell^{(1)}(M) \subset \cdots \subset \mathbb{C}\ell^{(2n)}(M) = \mathbb{C}\ell(M). \tag{3.44}$$

It is this filtration that defines the Getzler rescaling.

From Chapter 2, the heat kernel of $\eth^2 \in \mathrm{Diff}^2(M;\mathrm{S})$ is a distributional section of $\mathrm{END}(\mathrm{S})$ on the heat space M^2_H. Let us recall what this means exactly. Over $M \times M$, the bundle $\mathrm{END}(\mathrm{S})$ is the bundle with fiber $\mathrm{Hom}(\mathrm{S}_y, \mathrm{S}_x)$ at $(x,y) \in M^2$. Over the diagonal, $M_{\mathrm{diag}} \subset M^2$, we have a canonical identification

$$\mathrm{END}(\mathrm{S})|_{M_{\mathrm{diag}}} \cong \mathrm{End}(\mathrm{S}) \cong \mathbb{C}\ell(M), \tag{3.45}$$

but if $x \neq y$, $\mathrm{Hom}(\mathrm{S}_y, \mathrm{S}_x)$ is not an algebra, and it is does not make sense in general to identify it with $\mathbb{C}\ell(M)$.

The bundle $\mathrm{END}(\mathrm{S})$ is pulled back to M^2_H by the composite map

$$\pi_{M^2} : M^2_H \overset{\beta_H}{\to} \mathbb{R}_+ \times M^2 \to M^2$$

and we often drop the $\pi^*_{M^2}$ from the notation, simply writing $\mathrm{END}(\mathrm{S}) \to M^2_H$.

The identification (3.45) carries over to the preimage of the diagonal,

$$\pi^{-1}_{M^2}(M_{\mathrm{diag}}) = \mathbb{R}_+ \times M_{\mathrm{diag}} \cup \mathsf{hf} \subset M^2_H,$$

which is the union of the lifted diagonal (defined as the closure in M^2_H of the preimage under β_H of $(0,\infty) \times M_{\mathrm{diag}} \subset \mathbb{R}_+ \times M^2$) and the heat face.

Getzler's rescaling will be defined as the rescaling of $\mathrm{END}(\mathrm{S})$ at the heat face $\mathsf{hf} \subset M^2_H$ associated to the filtration (3.44). The only problem is, if you'll recall, we require the filtration to be defined *in a neighborhood* of hf, whereas at the moment it is only defined *at* hf. We need a way to extend the filtration off of hf.

One way to do this is by parallel transport. If we pick a vector field ν defined near hf which is normal to hf, then there is a sufficiently small neighborhood $\mathsf{hf} \times [0,\varepsilon)_x$ over which the bundle may be identified with the pullback from $\mathsf{hf} \times \{0\}$. Indeed, the condition $\nabla_\nu v = 0$ may be viewed as a family of ODE along the integral curves of ν near hf, which admit solutions for uniform ε by compactness of hf. The collar neighborhood is defined by $\mathsf{hf} \times [0,\varepsilon)_x$, where x satisfies $\nu(x) = 1$, and the parallel transport condition can then be stated as the property that

$$\nabla_\nu = \partial_x \tag{3.46}$$

on this neighborhood. In light of (3.46), notice that the sections with which we define the rescaled bundle, namely

$$v = v_0 + x v_1 + \cdots + x^{2n} v_{2n} + \mathcal{O}(x^{2n+1}), \quad v_j \in C^\infty(\mathsf{hf}; \mathbb{C}\ell^{(j)})$$

can be equivalently defined in terms of the condition

$$\nabla^j_\nu v|_{\mathsf{hf}} \in C^\infty(\mathsf{hf}; \mathbb{C}\ell^{(j)}), \quad 0 \leq j \leq 2n.$$

The rescaled bundle we obtain will depend on our choice of normal vector field, so we would like to make a good choice.

Recall that in Chapter 2, we were already filtering sections by order with respect to the function $\tau = t^{1/2}$ on M^2_H. Indeed, we defined Φ^k as the space $\tau^{-k}\rho_{\mathsf{tf}}^{\infty}C^{\infty}\big(M^2_H;\mathrm{END(S)}\big)$, so it would be nice to choose ν so that

$$\nu(\tau) = 1. \tag{3.47}$$

The one caveat here is that τ is not actually a boundary defining function for hf; indeed, $\tau = \rho_{\mathsf{tf}}\rho_{\mathsf{hf}}$ is the product of boundary defining functions for hf and tf, so any vector field satisfying $\nu(\tau) = 1$ will be singular at the corner $\mathsf{hf} \cap \mathsf{tf}$, where it will have the schematic form $s\rho_{\mathsf{tf}}^{-1}\partial_{\rho_{\mathsf{hf}}} + (1-s)\rho_{\mathsf{hf}}^{-1}\partial_{\rho_{\mathsf{tf}}}$ for some $s \in [0,1]$. However this is OK for our purposes, since we will always be applying it to the spaces $\rho_{\mathsf{tf}}^{\infty}\tau^{-k}C^{\infty} = \rho_{\mathsf{tf}}^{\infty}\rho_{\mathsf{hf}}^{-k}C^{\infty}$, on which the singular factor of ρ_{tf} may be absorbed, and on which it has the desired behavior of mapping $\rho_{\mathsf{tf}}^{\infty}\rho_{\mathsf{hf}}^{-k}C^{\infty}$ into $\rho_{\mathsf{tf}}^{\infty}\rho_{\mathsf{hf}}^{-k-1}C^{\infty} = \rho_{\mathsf{tf}}^{\infty}\tau^{-k-1}C^{\infty}$, i.e., of lowering the degree of growth/vanishing at hf by one.

Another condition we may impose, which will fix $\nu|_{\mathsf{hf}}$, is how it pushes forward under the blow-down map $\beta : M^2_H \to \mathbb{R}_+ \times M^2$. Under β, a point in hf is mapped to a point in $\{0\} \times M_{\mathrm{diag}} \subset (\mathbb{R}_+)_\tau \times M^2$, and any normal vector at that point will be sent to a normal vector at the image. There are various natural choices for a complementary subbundle to $T(\{0\} \times M_{\mathrm{diag}})$; for instance, we can choose to identify $N(\{0\} \times M_{\mathrm{diag}})$ with the subbundle[15]

$$N(\{0\} \times M_{\mathrm{diag}}) \cong \big\{(\eta, \zeta, 0)\big\} \subset T(\mathbb{R}_+ \times M^2)|_{\{0\}\times M_{\mathrm{diag}}}$$

Recalling that we also have a canonical identification $\mathsf{hf} \cong \overline{TM}$, it makes sense to impose the condition that

$$d(\beta_H)_{(x,\zeta)}(\nu) = (1, \zeta, 0) \in T_{(0,x,x)}(\mathbb{R}_+ \times M^2), \quad \forall (x,\zeta) \in \mathring{\mathsf{hf}} \cong TM \tag{3.48}$$

Here we are using ∂_τ to trivialize $T(\mathbb{R}_+)_\tau$, so the coefficient 1 denotes the vector $1\partial_\tau$. This formula is important, so take a moment to digest its meaning. We are saying that, at the point (x, ζ) in TM, which we have identified with $\mathring{\mathsf{hf}}$, ν should come from the normal vector $(1, \zeta, 0) \in T(\mathbb{R}_+)_\tau \oplus TM \oplus TM$ to the time zero diagonal.

Proposition 3.38. *There exists a vector field ν, defined in a collar neighborhood of $\mathring{\mathsf{hf}} \subset M^2_H$ which satisfies* (3.47) *and* (3.48)*, and extends to a singular vector field on a neighborhood of* hf.

Proof. It is useful to employ a slightly different local coordinate convention for the blow-up $M^2_H \to \mathbb{R}_+ \times M^2$ than before; namely, if (τ, x, y) are local coordinates near the diagonal on the latter space, with $x = \pi_L^* z$ and $y = \pi_R^* z$ given by pulling back the same local coordinates on M from the right and left, define coordinates on M^2_H by

$$(\tau, \zeta, y) = \big(\tau, (x-y)/\tau, y\big). \tag{3.49}$$

[15] Various other natural choices exist, for instance the subbundle $\big\{(\eta, \frac{1}{2}\zeta, -\frac{1}{2}\zeta)\big\}$ or the subbundle $\{(\eta, 0, -\zeta)\}$. Either of these choices lead to the same formulas below.

Then the blow-down map is given by

$$\beta : (\tau, \zeta, y) \mapsto (\tau, \tau\zeta + y, y)$$

and the tangent vector ∂_τ upstairs at a point $(0, \zeta, y)$ is mapped to

$$d(\beta_H)_{(0,\zeta,y)}(\partial_\tau) = \partial_\tau + \zeta \cdot \partial_x, \tag{3.50}$$

which has the required form.

To see that this is independent of the choice of local coordinates, observe that (3.50), multiplied by τ, is the vector field

$$\tau\partial_\tau + x \cdot \partial_x,$$

whose restriction to the diagonal $\mathbb{R}_+ \times M_{\text{diag}}$ is independent of the choice of coordinates. Thus we can take ν to be the lift of this, divided by τ, and by reversing the computations above, this has the form ∂_τ in local coordinates upstairs defined by the convention (3.49). $\square$

Definition 3.39. The **Getzler rescaling** of END(S) at $\mathsf{hf} \subset M_H^2$ is the vector bundle ${}^{\mathrm{G}}\mathrm{END(S)} \to M_H^2$ defined by the property

$$C^\infty\big(M_H^2; {}^{\mathrm{G}}\mathrm{END(S)}\big) = \big\{A : \nabla_\nu^j A|_{\mathsf{hf}} \in C^\infty\big(\mathsf{hf}; \mathbb{C}\ell^{(j)}(M)\big),\ 0 \leq j \leq 2n\big\} \subset C^\infty\big(M_H^2; \mathrm{END(S)}\big).$$

Here we employ the isomorphisms $\mathrm{END(S)}|_{\mathsf{hf}} \cong \mathrm{End(S)} \cong \mathbb{C}\ell(M)$, the connection ∇ is the Levi-Civita connection on $\mathrm{END(S)} = \pi_{L,M}^* \mathrm{S} \boxtimes \pi_{R,M}^* \mathrm{S}^*$, and the (singular) normal vector field ν satisfies (3.47) and (3.48). In particular, a section of ${}^{\mathrm{G}}\mathrm{END(S)}$, viewed as a section of the unrescaled bundle END(S), has the form

$$B = B_0 + \tau B_1 + \cdots + \tau^{2n} B_{2n} + \mathcal{O}(\tau^{2n+1}), \quad B_j \in C^\infty\big(\mathsf{hf}; \mathbb{C}\ell^{(j)}(M)\big) \tag{3.51}$$

at $\mathring{\mathsf{hf}} \subset M_H^2$. From Proposition 3.37, there is a canonical isomorphism

$${}^{\mathrm{G}}\mathrm{END(S)}|_{\mathsf{hf}} \cong \mathrm{Gr}\, \mathbb{C}\ell(M) \cong \Lambda M, \tag{3.52}$$

under which the image of a section (3.51) is identified with

$$B|_{\mathsf{hf}} = ([B_0], [B_1], \ldots, [B_{2n}]) \in C^\infty(\mathsf{hf}; \Lambda^0 M \oplus \cdots \oplus \Lambda^{2n} M).$$

We define the **rescaled kernel spaces** by

$${}^{\mathrm{G}}\Phi^k = \tau^{-k} \rho_{\mathsf{tf}}^\infty C^\infty\big(M_H^2; {}^{\mathrm{G}}\mathrm{END(S)}\big), \quad k \in \mathbb{Z}.$$

Evidently ${}^{\mathrm{G}}\Phi^k \subset {}^{\mathrm{G}}\Phi^l$ for $k \leq l$, and from the bundle map ${}^{\mathrm{G}}\mathrm{END(S)} \to \mathrm{END(S)}$, there is an inclusion

$${}^{\mathrm{G}}\Phi^k \subset \Phi^k \quad \forall\ k, \tag{3.53}$$

of the rescaled kernel spaces into the unrescaled kernel spaces. If $A \in {}^{\mathrm{G}}\Phi^k$, then its image in Φ^k has the form

$$\begin{gathered} A = \tau^{-k} A_0 + \tau^{1-k} A_1 + \cdots + \tau^{2n-k} A_{2n} + \mathcal{O}(\tau^{2n+1}) \in \Phi^k, \\ A_j = \tfrac{1}{j!} \nabla_\nu^j (\tau^k A)|_{\mathsf{hf}} \in \mathcal{S}\big(TM; \mathbb{C}\ell^{(j)}(M)\big), \end{gathered} \tag{3.54}$$

where we use the identification $\rho_{\mathsf{tf}}^\infty C^\infty(\mathsf{hf}) \cong \mathcal{S}(TM)$.

By analogy with (2.15), we define the **rescaled heat model operator** of $A \in {}^{\mathrm{G}}\Phi^k$ by

$${}^{\mathrm{G}}N(A) = {}^{\mathrm{G}}N_k(A) = (\tau^k A)|_{\mathsf{hf}} \in \mathcal{S}(TM; \Lambda M), \quad A \in {}^{\mathrm{G}}\Phi^k.$$

Here we use the isomorphism (3.52). In terms of (3.54) the normal operator takes the form

$${}^{\mathrm{G}}N(A) = ([A_0], [A_1], \ldots, [A_{2n}]) \in \mathcal{S}\big(TM; \Lambda^0 M \oplus \cdots \oplus \Lambda^{2n} M\big).$$

For $A \in {}^{\mathrm{G}}\Phi^k$, it follows from (3.53) that we may compare the rescaled model operator with the unrescaled one, and from the expression (3.54) it is clear that

$$N_k(A) = \big[{}^{\mathrm{G}}N_k(A)\big]_0 = [A_0] \in \mathcal{S}(TM; \Lambda^0 M),$$

where $\big[\,\cdot\,\big]_l : \Lambda M \to \Lambda^l M$ denotes the projection onto the l-form component of a total form, for $0 \leq l \leq 2n$.

From Proposition 2.6 and Corollary 2.7, we obtain the following result.

Proposition 3.40. *Let $A \in {}^{\mathrm{G}}\Phi^k$. Then A defines an operator*

$$A : C^\infty(M; \mathrm{S}) \to t^{(2n-k)/2} C^\infty((\mathbb{R}_+)_{1/2} \times M; \mathrm{S}) \subset t^{(2n-k)/2} C^0(\mathbb{R}_+ \times M; \mathrm{S}).$$

If $k = 2n$, then the time 0 restriction of A is the operator

$$A|_{t=0} : C^\infty(M; \mathrm{S}) \to C^\infty(M; \mathrm{S}), \quad A|_{t=0} u = \Big(\int_{\mathrm{fib}} \big[{}^{\mathrm{G}}N(A)\big]_0 \Big) u.$$

The next result, which is a direct analogue of Proposition 2.22 from our earlier heat kernel construction, shows that Getzler's rescaling fulfills the promise of encoding the short-time limit of the supertrace of a heat kernel A in terms of the leading order term, ${}^{\mathrm{G}}N(A)$, in the expansion at hf.

Proposition 3.41. *If $A \in {}^{\mathrm{G}}\Phi^{2n}$, then* $\operatorname{Str} A$ *has a complete* **short time asymptotic expansion** *as $t \searrow 0$ of the form*

$$\operatorname{Str} A \sim \sum_{j=0}^\infty a_j \, t^{j/2},$$

with $a_0 = \int_M \operatorname{str} \big[{}^{\mathrm{G}}N(A)(0,x)\big]_{2n} \, \mathrm{dVol}_g$.

Proof. We may view A as a section of Φ^k of the form (3.54), and then from the discussion in §3.3 it follows that, *a priori*

$$\operatorname{Str} a \sim t^{-n} \sum_{j=0}^{\infty} \widetilde{a}_j t^{j/2}, \quad \widetilde{a}_j = \int_M \operatorname{str} A_j(x,0)\, \mathrm{dVol}_g.$$

However, since $A_j \in \mathcal{S}\big(TM; \mathbb{C}\ell^{(j)}(M)\big)$, for $j \leq 2n$ and $\operatorname{str} \equiv 0$ on $\mathbb{C}\ell^{(j)}(M)$ by Proposition 3.32, it follows that $\widetilde{a}_j = 0$ for $j < 2n$. Reindexing and noting that $\operatorname{str} A_{2n} = \operatorname{str}[A_{2n}] = \operatorname{str}\big[{}^{\mathrm{G}}N(A)\big]_{2n}$ proves the claim. $\square$

The cost of working with the rescaled bundle is that we change the model PDE to be solved over $\mathring{\mathsf{hf}} \cong TM$ in the iterative construction. Recall that in §2.3.4 we computed the action on $N(A)$ of the lift of vector fields of the form τV, where V was a vector field on M, pulled back to $\mathbb{R}_+ \times M^2$ from the leftmost factor of M. We do the analogous computation here for the covariant derivative $\nabla_{\tau V}$, making use of the defining condition $\nabla_\nu^j u|_{\mathsf{hf}} \in C^\infty\big(\mathsf{hf}; \mathbb{C}\ell^{(j)}(M)\big)$ for sections of ${}^{\mathrm{G}}\mathrm{END}(\mathrm{S})$.

Lemma 3.42. *Let $A \in {}^{\mathrm{G}}\Phi^k$ and let V be the pullback of a vector field on M to $\mathbb{R}_+ \times M^2$ from the left factor of M. Then*

$${}^{\mathrm{G}}N(\nabla_{\tau V} A) = \big(\sigma(V)_\zeta + \tfrac{1}{4} R^\wedge_{\zeta,V}\big)\, {}^{\mathrm{G}}N(A),$$

where $\sigma(V)_\zeta \in \mathrm{Diff}^1_{\mathrm{fib}}(TM)$ is the symbol of V, considered as a fiberwise (scalar) differential operator on TM, and

$$R^\wedge_{\zeta,V} = \sum_{i<j} \langle R_{\zeta,V}(e_i), e_j\rangle (e_i \wedge e_j) : \Lambda^k M \to \Lambda^{k+2} M.$$

More explicitly, if ${}^{\mathrm{G}}N(A) = (A_0, A_1, \dots, A_{2n}) \in \mathcal{S}(TM; \Lambda M)$, then

$$\begin{gathered} {}^{\mathrm{G}}N(\nabla_{\tau V} A) = (B_0, \dots, B_{2n}), \\ B_j(x,\zeta) = \sigma(V)_\zeta A_j(x,\zeta) + \tfrac{1}{4} R_{\zeta,V} A_{j-2}(x,\zeta), \quad 0 \leq j \leq 2n. \end{gathered} \tag{3.55}$$

Remark. The reason the computation is more complicated than in §2.3.4 is that we can no longer simply disregard all terms of order $\mathcal{O}(\tau)$ at hf, since terms of various orders in τ are baked into the definition of sections of ${}^{\mathrm{G}}\mathrm{END}(\mathrm{S})$.

Proof. We are using the given (Levi-Civita) connection on $\mathrm{END}(\mathrm{S})$, which is the lift of a connection from M^2 to M^2_H. This has several important implications. First of all, under the identification $\mathrm{End}(\mathrm{S}) \cong \mathbb{C}\ell(M)$, the restriction of the connection to hf preserves the filtration $\bigcup_j \mathbb{C}\ell^{(j)}(M)$; in particular this holds for any covariant derivatives taken along vector fields tangent to hf. Next, since the curvature of the connection is a(n endomorphism-valued) 2-form, it commutes with pullback in the sense that

$$R_{X,Y} := (\pi^*_{M^2} R)_{X,Y} = \pi^*_{M^2}(R_{(\pi_{M^2})_* X, (\pi_{M^2})_* Y}).$$

In particular, while the lift of V itself is singular at hf (hence we lift τV instead), the curvature term $R_{\nu,V}$ is bounded at hf and $R_{\nu,\tau V} = \tau R_{\nu,V}$ vanishes.

We fix an element $A \in {}^{\mathrm{G}}\Phi^k$ and regard it for the purposes of the computation as a section of the unrescaled bundle END(S), of the form (3.54). After the computation we will reinterpret the result as a section of the rescaled bundle ${}^{\mathrm{G}}$END(S). Commuting an overall factor of τ^k through the computation will have no affect, so it suffices to consider the case that $A \in {}^{\mathrm{G}}\Phi^0$. Thus, regarded as a section of END(S),

$$A = A_0 + \tau A_1 + \cdots + \tau^{2n} A_{2n} + \mathcal{O}(\tau^{2n+1}), \quad A_j = \tfrac{1}{j!}\nabla_\nu^j A|_{\mathsf{hf}} \in \mathcal{S}\big(TM; \mathbb{C}\ell^{(j)}(M)\big),$$
$${}^{\mathrm{G}}N(A) = \bigoplus_j \big[A_j|_{\mathsf{hf}}\big]_j = \bigoplus_j \big[\tfrac{1}{j!}\nabla_\nu^j A|_{\mathsf{hf}}\big]_j \in \mathcal{S}(TM; \Lambda^0 M \oplus \cdots \oplus \Lambda^{2n} M).$$

Here and below, we use the notation $\big[\,\cdot\,\big]_j$ to denote the associated graded projection from $\mathbb{C}\ell(M) \cong \Lambda M$ onto $\Lambda^j M \cong \mathbb{C}\ell^{(j)}(M)/\mathbb{C}\ell^{(j-1)}(M)$. We wish to compute ${}^{\mathrm{G}}N(\nabla_{\tau V} A)$, which will involve commuting $\nabla_{\tau V}$ with ∇_ν^j. For $j = 0$, the computation is easy, and the 0-form component of ${}^{\mathrm{G}}N(\nabla_{\tau V} A)$ is simply

$$\big[{}^{\mathrm{G}}N(\nabla_{\tau V} A)\big]_0 = \big[\nabla_{\tau V} A|_{\mathsf{hf}}\big]_0 = \nabla_{\tau V}|_{\mathsf{hf}} A_0,$$

since $\nabla_{\tau V}$ preserves the filtration.

For $j = 1$, we have

$$[{}^{\mathrm{G}}N(\nabla_{\tau V} A)]_1 = \big[\nabla_\nu \nabla_{\tau V} A|_{\mathsf{hf}}\big]_1, \tag{3.56}$$

and we make use of the curvature identity $\nabla_\nu \nabla_{\tau V} = \nabla_{\tau V}\nabla_\nu + R_{\nu,\tau V} + \nabla_{[\nu,\tau V]}$. As noted above, $R_{\nu,\tau V} = \tau R_{\nu,V}$, so this term will vanish when restricted to hf. For the last term, we claim that $[\nu, \tau V]$ is a vector field tangent to hf. This may be verified in local coordinates (3.49). Indeed, if V has the local form $V = \sum_j a_j(x)\partial_{x_j}$, then τV lifts to $\sum_j a_j(y + \tau\zeta)\partial_{\zeta_j}$ and $[\nu, \tau V]$ is a sum of various derivatives of the a_j times the ∂_{ζ_j}. In particular $\nabla_{[\nu,\tau V]}$ preserves the filtration degree of $\mathbb{C}\ell(M)$. Thus (3.56) becomes

$$\begin{aligned}
\big[{}^{\mathrm{G}}N(\nabla_{\tau V} A)\big]_1 &= \big[\nabla_{\tau V}\nabla_\nu A|_{\mathsf{hf}} + \tau R_{\nu,V} A|_{\mathsf{hf}} + \nabla_{[\nu,\tau V]} A|_{\mathsf{hf}}\big]_1 \\
&= \big[\nabla_{\tau V}|_{\mathsf{hf}} A_1\big]_1 + \big[\nabla_{[\nu,\tau V]}|_{\mathsf{hf}} A_0\big]_1 \\
&= \big[\nabla_{\tau V}|_{\mathsf{hf}} A_1\big]_1.
\end{aligned}$$

Here we have used the fact that $\nabla_{[\nu,\tau V]}|_{\mathsf{hf}} A_0$ is a section of $\mathbb{C}\ell^{(0)}(M)$, hence vanishes when we take the quotient in $\Lambda^1 M \cong \mathbb{C}\ell^{(1)}/\mathbb{C}\ell^{(0)}$.

Subsequent normal derivatives generate increasingly many terms, but we can simplify the computation by distinguishing between *relevant* terms, which may contribute later on, from *irrelevant* terms, which will never contribute anything. At each step we have an expansion at hf in terms of the form $\tau^k \mathbb{C}\ell^{(l)}$, with the expansion at the next step determined by taking ∇_ν, which lowers degree in τ. To compute the contribution to the normal operator at the jth step, we apply the operations $\cdot|_{\mathsf{hf}}$ and $[\cdot]_j$ for some j; the first operation kills all terms of order greater than 0 in τ, and the second operation kills all remaining terms of order less than j in $\mathbb{C}\ell^{(\cdot)}$. Thus in the first step we have

$$\nabla_\nu \nabla_{\tau V} A = \underbrace{\nabla_{\tau V} \nabla_\nu A}_{\mathcal{O}(\tau^0 \mathbb{C}\ell^{(1)})} + \underbrace{\tau R_{\nu,V} A}_{\mathcal{O}(\tau^1 \mathbb{C}\ell^{(2)})} + \underbrace{\nabla_{[\nu,\tau V]} A}_{\mathcal{O}(\tau^0 \mathbb{C}\ell^{(0)})}.$$

Only the first term contributes to $\left[{}^{\mathrm{G}}N(\nabla_{\tau V} A)\right]_1$, but the second term is still relevant, since in the second step ∇_ν will map this into $\tau^0 \mathbb{C}\ell^{(2)}$, which will contribute to $\left[{}^{\mathrm{G}}N(\nabla_{\tau V} A)\right]_2$. In contrast, the third term is irrelevant since even after j steps, we will have applied ∇_ν^j, resulting in terms of the form $\tau^0 \mathbb{C}\ell^{(j)}$, which will vanish on taking $\left[\,\cdot\,\right]_{j+1}$. In general, terms of the form $\tau^j \mathbb{C}\ell^{(l)}$ will be relevant at step k if and only if $l - j \geq k$.

This is the basis for an induction, where we may suppose

$$\nabla_\nu^k \nabla_{\tau V} A = \nabla_{\tau V} \nabla_\nu^k A + k\, \tau R_{\nu,V} \nabla_\nu^{k-1} A + \tfrac{k(k-1)}{2} R_{\nu,V} \nabla_\nu^{k-2} A + \text{irrel.}$$

Then recalling that $A_k = \frac{1}{k!} \nabla_\nu^k A|_{\mathsf{hf}}$, we have

$$\left[{}^{\mathrm{G}}N(\nabla_{\tau V} A)\right]_k = \left[\tfrac{1}{k!} \nabla_\nu^k \nabla_{\tau V} A|_{\mathsf{hf}}\right]_k = \left[\nabla_{\tau V}|_{\mathsf{hf}} A_k + \tfrac{1}{2} R_{\nu,V} A_{k-2}\right]_k. \tag{3.57}$$

Here we are using $\frac{k(k-1)}{2} \frac{1}{k!} = \frac{1}{2} \frac{1}{(k-2)!}$. To complete the induction we compute

$$\begin{aligned}
\nabla_\nu^{k+1} \nabla_{\tau V} A &= \nabla_\nu \big(\nabla_{\tau V} \nabla_\nu^k A + k\tau R_{\nu,V} \nabla_\nu^{k-1} A + \tfrac{k(k-1)}{2} R_{\nu,V} \nabla_\nu^{k-2} A\big) + \text{irrel.} \\
&= \underbrace{\nabla_{\tau V} \nabla_\nu^{k+1} A}_{\mathcal{O}(\tau^0 \mathbb{C}\ell^{(k+1)})} + \underbrace{\tau R_{\nu,V} \nabla_\nu^k A}_{\mathcal{O}(\tau^1 \mathbb{C}\ell^{(k+2)})} + \underbrace{\nabla_{[\nu,\tau V]} \nabla_\nu^k A}_{\mathcal{O}(\tau^0 \mathbb{C}\ell^{(k)})} \\
&\quad + \underbrace{k R_{\nu,V} \nabla_\nu^{k-1} A}_{\mathcal{O}(\tau^0 \mathbb{C}\ell^{(k+1)})} + \underbrace{\tau k (\nabla_\nu R_{\nu,V}) \nabla_\nu^{k-1} A}_{\mathcal{O}(\tau^1 \mathbb{C}\ell^{(k+1)})} + \underbrace{k\tau R_{\nu,V} \nabla_\nu^k A}_{\mathcal{O}(\tau^1 \mathbb{C}\ell^{(k+2)})} \\
&\quad + \underbrace{\tfrac{k(k-1)}{2} (\nabla_\nu R_{\nu,V}) \nabla_\nu^{k-2} A}_{\mathcal{O}(\tau^0 \mathbb{C}\ell^{(k)})} + \underbrace{\tfrac{k(k-1)}{2} R_{\nu,V} \nabla_\nu^{k-1} A}_{\mathcal{O}(\tau^0 \mathbb{C}\ell^{(k+1)})} + \text{irrel.} \\
&= \nabla_{\tau V} \nabla_\nu^{k+1} A + (k+1)\tau R_{\nu,V} \nabla_\nu^k A + \tfrac{k(k+1)}{2} R_{\nu,V} \nabla_\nu^{k-1} A + \text{irrel.}
\end{aligned}$$

We have used the identity $\nabla_\nu(\tau u) = u + \tau \nabla_\nu u$ (which follows from $\nu(\tau) = 1$) and have discarded all terms of order $\mathcal{O}(\tau^j \mathbb{C}\ell^{(l)})$ with $l - j < k + 1$ as irrelevant.

It remains to relate (3.57) with the claimed formula (3.55). Working in a local coordinate frame $\{e_1, \ldots, e_{2n}\} = \{\partial_{x_1}, \ldots, \partial_{x_{2n}}\}$ downstairs (which we may assume is orthonormal at a given point), we express the action of $\nabla_{\tau V}$ on $\mathbb{C}\ell(M) \cong \Lambda M$ as (c.f. (3.26))

$$\begin{aligned}
\nabla_{\tau V} &= \tau \textstyle\sum_i a_i(x) \big(\partial_{x_i} + \frac{1}{2} \sum_{j<k} {\Gamma^k}_{ij} \mathrm{c}\ell(e_j e_k)\big), \\
&\cong \textstyle\sum_i a_i(x) (\tau \partial_{x_i} + \frac{1}{2} \sum_{j<k} \tau {\Gamma^k}_{ij} (e_j \wedge e_k))
\end{aligned}$$

This lifts near $\mathsf{hf} \subset M_H^2$ to

$$\nabla_{\tau V} = \sum_i a_i(x) \partial_{\zeta_i} + \mathcal{O}(\tau) = \sigma(V)_\zeta + \mathcal{O}(\tau),$$

hence $\nabla_{\tau V}|_{\mathsf{hf}} = \sigma(V)_\zeta$.

For the curvature term, we again use the identification $\mathrm{Gr}\,\mathbb{C}\ell(M) \cong \Lambda M$ to write

$$\frac{1}{2}R_{\nu,V} = \frac{1}{4}\sum_{j<k} \langle R_{\nu,V}(e_j), e_k\rangle (e_j \wedge e_k) : \Lambda^j M \to \Lambda^{j+2} M.$$

In addition, we recall that $(d\beta)_{(x,\zeta)}(\nu) = (1,\zeta,0) = \partial_\tau + \zeta\cdot\partial_x$. Then since $R_{\nu,V} = R_{(\pi_{M^2})_*\nu,(\pi_{M^2})_*V}$ as the connection is pulled back from M^2 to M^2_H via $\pi_{M^2}\circ\beta : M^2_H \to M^2$—in particular, $(\pi_{M^2})_*\nu = (\zeta, 0)$—we conclude

$$\frac{1}{2}R_{\nu,V}(x,\zeta) = \frac{1}{4}\sum_{k<l} \langle R(x)_{\zeta,V}(e_k), e_l\rangle (e_k \wedge e_l) : \Lambda^j M \to \Lambda^{j+2} M. \qquad \square$$

Now, R is naturally a 2-form with values in endomorphisms of TM (or an endomorphism of TM with coefficients in 2-forms), and in the above expression we are using the endomorphism part to act as a 2-form, which is a bit unnatural. By the symmetry properties of the curvature tensor, this can be remedied. For a local orthonormal frame, define

$$\mathsf{R}_{ij} = \langle R(e_i), e_j\rangle = \sum_{k<l} \langle R_{e_k,e_l}(e_i), e_j\rangle (e_k \wedge e_l) \in C^\infty(M;\Lambda^2).$$

The $\mathsf{R}_{ij} = \mathsf{R}(x)_{ij}$ are the components of a local family of skew-adjoint matrices of 2-forms, and setting $V = e_i$ above, we can write

$$\frac{1}{4}R_{\zeta,e_i} = -\frac{1}{4}\sum_{j<k}\big\langle R_{e_j,e_k}(e_i), \zeta\big\rangle (e_j \wedge e_k) = -\frac{1}{4}\sum_j \mathsf{R}_{ij}\zeta_j \in C^\infty(TM;\Lambda^2). \tag{3.58}$$

Proposition 3.43. *Let $A \in {}^{\mathrm{G}}\Phi^k$. Then*

$${}^{\mathrm{G}}N(t\eth^2 A) = H_\zeta\,{}^{\mathrm{G}}N(A),$$

where H_ζ is the fiberwise operator

$$H_\zeta = -(\textstyle\sum_i \partial_{\zeta_i} - \frac{1}{4}\sum_j \mathsf{R}_{ij}\zeta_j)^2. \tag{3.59}$$

More generally, let $\eth_F$ be the spin Dirac operator twisted by a bundle $F \to M$ with connection ∇^F and curvature K^F. Then

$${}^{\mathrm{G}}N\big(t\eth_F^2)A\big) = \big(H_\zeta + K^F\big)\,{}^{\mathrm{G}}N(A),$$

where $K^F = \sum_{i<j} K^F(e_i,e_j)(e_i \wedge e_j) : \Lambda^* M \otimes F \to \Lambda^{*+2} M \otimes F$.

Proof. We need only use the twisted Lichnerowicz formula and

$$t\eth_F^2 = \tau^2\eth_F^2 = -\sum_i \nabla_{\tau e_i}\nabla_{\tau e_i} + \frac{\tau^2}{4}\kappa + \tau^2\mathcal{K}, \quad \tau^2\mathcal{K} = \sum_{i<j} K^F(e_i,e_j)\mathrm{c}\ell(\tau e_i\,\tau e_j).$$

Lifted to $\mathsf{hf} \subset M^2_H$, the term $\tau^2/4\kappa$ vanishes, and applying Lemma 3.42 and (3.58) to the first term gives H_ζ, so it remains to determine how the last term acts on the rescaled bundle. If

$$A = A_0 + \tau A_1 + \cdots + \tau^{2n} A_{2n} + \mathcal{O}(\tau^{2n+1}),$$

then clearly $\mathrm{c}\ell(\tau e_i) = \tau \mathrm{c}\ell(e_i)$ maps $\tau^j A_j$ to $\tau^{j+1} e_i \cdot A_j$. Under the identification of ${}^{\mathrm{G}}\mathrm{END}(\mathrm{S})$ with ΛM, this is precisely the operation $e_i \wedge \cdot : \Lambda^j M \to \Lambda^{j+1}$. □

The computation of the action of $t\partial_t = \frac{1}{2}\tau\partial_\tau$ on rescaled normal operators is the same as the unrescaled case, namely

$${}^{\mathrm{G}}N(t\partial_t A) = -\tfrac{1}{2}(\zeta \cdot \partial_\zeta + 2n - k)\,{}^{\mathrm{G}}N(A), \quad A \in {}^{\mathrm{G}}\Phi^k.$$

Thus the full action of the heat operator $t(\partial_t + \eth^2_F)$ on $A \in {}^{\mathrm{G}}\Phi^k$ is given by

$${}^{\mathrm{G}}N\big(t(\partial_t + \eth^2_F)A\big) = \big(-\tfrac{1}{2}(\zeta \cdot \partial_\zeta + 2n - k) + H_\zeta + K^F \big)\,{}^{\mathrm{G}}N(A).$$

However, rather than solve the model equation on TM explicitly as we did in §2, we will instead take a slightly more classical approach and study the heat equation $(\partial_t + H_\zeta)u = 0$ for operators of the form (3.59). The trick here is to note that if we freeze the coefficients $\mathsf{R}(x)_{ij}$ at a point $x \in M$, then we can study the heat operator

$$\partial_t - \textstyle\sum_i \big(\partial_{\zeta_i} - \frac{1}{4}\sum_j \mathsf{R}_{ij}\zeta_j\big)^2$$

on $\mathbb{R}^n$ with Euclidean coordinates ζ. Once we have determined a heat kernel for this operator, we can then lift it to the heat space $(\mathbb{R}^n)^2_H$, and its restriction to any fiber of $\mathsf{hf} \subset (\mathbb{R}^n)^2_H$ will give a solution to the model equation on the fiber of $\mathsf{hf} \subset M^2_H$ over x.

3.3.3 Mehler's formula

The starting point is the heat equation for the quantum harmonic oscillator on $\mathbb{R}$:

$$\begin{aligned}(\partial_t - \partial_x^2 + x^2)u(t,x,y) &= 0,\\ u(0,x,y) &= \delta(x-y).\end{aligned}$$

This admits an explicit solution, known as **Mehler's formula**:

$$u(t,x,y) = \big(2\pi \sinh(2t)\big)^{-1/2} \exp\big(-\tfrac{1}{2}\coth(2t)(x^2+y^2) + \operatorname{cosech}(2t)(xy)\big), \tag{3.60}$$

Exercise 3.5. Derive (3.60) starting from the ansatz $u(t,x,y) = \exp\big(a_t/2(x^2+y^2) + b_t xy + c_t\big)$, which leads to explicitly solvable ODEs for a_t, b_t and c_t. This ansatz is justified by the fact that the operator is quadratic in x (both in its differential and potential terms), the fact that the heat kernel should be symmetric in x and y, and the desire for u to be a family of Gaussians.

We may make this appear a bit more like the heat kernel for the classical Laplacian, and combining this with the change of variables $t \mapsto at$, $x \mapsto a^{1/2}x$, $y \mapsto a^{1/2}y$ gives the heat kernel

$$u_a(t,x,y) = \frac{1}{(4\pi t)^{1/2}}\Big(\frac{2at}{\sinh 2at}\Big)^{1/2} \exp\Big(-\frac{1}{4t}\frac{2at}{\sinh 2at}\big(\cosh 2at(x^2+y^2) - 2xy\big)\Big)\exp(-tf).$$

which is the fundamental solution to

$$(\partial_t - \partial_x^2 + a^2x^2 + f)u_a = 0, \quad f \in \mathbb{R}.$$

It is convenient to set $y = 0$ and just work with $p_a(t,x) = u_a(t,x,0)$, which we shall do from now on.

Now suppose A_{ij} is an $n \times n$ antisymmetric matrix with coefficients in a nilpotent, commutative algebra $\mathcal{A}$. In practice we are interested in the case that $\mathcal{A} = \Lambda^{\mathrm{even}}\mathbb{R}^n$. Since $\mathcal{A}$ is nilpotent, we may apply arbitrary power series to A_{ij}, the result of which is always another matrix with coefficients in $\mathcal{A}$. Let $K \in \mathrm{End}(W)$ be an arbitrary endomorphism of some finite dimensional vector space W. We consider the heat operator

$$L := \partial_t - \Big(\textstyle\sum_{i=1}^n \partial_{x_i} - \sum_{j=1}^n A_{ij}x_j\Big)^2 + K \in \mathrm{Diff}^2(\mathbb{R}_+ \times \mathbb{R}^n; \mathcal{A}\otimes W). \tag{3.61}$$

Proposition 3.44. *The $\mathcal{A}\otimes\mathrm{End}(W)$-valued function*

$$P(t,x) = \frac{1}{(4\pi t)^{n/2}} \det\Big(\frac{2tA}{\sinh 2tA}\Big)^{1/2} \exp\big(-\tfrac{1}{4t}\langle 2tA\coth 2tA\, x, x\rangle\big)\otimes\exp(-tK)$$

is a solution to (3.61) *with* $P(0,x) = \delta(x)1\otimes I$.

Proof. Note that LP is analytic in the coefficients A_{ij}; since an analytic function on the algebra $\mathcal{A}$ is determined by its values on $\mathbb{R}\subset\mathcal{A}$ it suffices to verify the claim when A_{ij} are real. By making an orthogonal transformation in $\mathbb{R}^n$, we can assume A is a direct sum of 2×2 blocks, thus it suffices to prove the result in the case that $n = 2$ and $A = \begin{pmatrix} 0 & a \\ -a & 0\end{pmatrix}$, which we assume from now on. The contribution of K is also clear, so we shall assume $K = 0$ below to simplify the notation.

Note that $z/\sinh z$ and $z\coth z$ are even functions of z, which is to say they are given by power series in z^2. Since $A^2 = -a^2 I$, it follows that these even functions take the same value on A as on iaI. Thus

$$2tA\coth(2tA) = 2ita\coth(2ita)I = 2ita\cot(2ta)I,$$
$$\det\Big(\frac{2tA}{\sinh 2tA}\Big)^{1/2} = \det\Big(\frac{2ita}{\sinh 2ita}I\Big)^{1/2} = \det\Big(\frac{2ta}{\sin 2ta}I\Big)^{1/2} = \frac{2ta}{\sin 2ta},$$

and therefore P reduces to

$$P(t,x) = (4\pi t)^{-1}\frac{2ta}{\sin 2ta}\exp\big(-2ita\cot(2ta)(x_1^2+x_2^2)/4t\big).$$

On the other hand, expanding L out, we have

$$L = L' + V, \quad L' = \partial_t - (\partial_{x_1}^2 + \partial_{x_2}^2) - a^2(x_1^2 + x_2^2), \quad V = a(x_1\partial_{x_2} - x_2\partial_{x_1}).$$

Ignoring the last term for a moment and using Mehler's kernel as above with a replaced by ia, we see that a heat kernel for L' is given by

$$\begin{aligned} p_{ia}(t,x_1)p_{ia}(t,x_2) &= (4\pi t)^{-1/2}\Big(\frac{2ita}{\sinh 2ita}\Big)^{1/2}\exp(-2ita\coth(2ita)x_1^2/4t) \\ &\quad\times (4\pi t)^{-1/2}\Big(\frac{2ita}{\sinh 2ita}\Big)^{1/2}\exp(-2ita\coth(2ita)x_2^2/4t) \\ &= (4\pi t)^{-1}\Big(\frac{2ta}{\sin 2ta}\Big)\exp\big(-2ita\cot(2ta)(x_1^2+x_2^2)/4t\big) \end{aligned}$$

which is precisely $P(t,x)$. To complete the proof, note that $V = a(x_1\partial_{x_2} - x_2\partial_{x_1})$ is an infinitesimal rotation; in particular $V\,\|x\| = 0$, so it follows that $LP = 0$. □

3.4 The index theorem

We are interested in the generalized Mehler formula above in the case that $A_{ij} = \frac{1}{4}\mathsf{R}_{ij}$ is a two-form, with $\mathcal{A} = \Lambda^{\text{even}}\mathbb{R}^n$, and K is a two-form with coefficients in $\operatorname{End}(F)$. Lifting $P(t,x)$ to the Euclidean heat space $(\mathbb{R}^n)^2_H$ and applying the Getzler rescaling amounts to replacing R_{ij} by $t^{-1}\mathsf{R}_{ij}$, replacing K by $t^{-1}K$, and replacing $x/t^{1/2}$ by ζ. Removing an overall factor of $t^{-n/2}$, we obtain the function

$$(4\pi)^{-n/2}\det\Big(\frac{\mathsf{R}/2}{\sinh\mathsf{R}/2}\Big)^{1/2}\exp\big(-\tfrac{1}{4}\langle\mathsf{R}/2\coth(\mathsf{R}/2)\zeta,\zeta\rangle\big)\exp(-K). \tag{3.62}$$

Theorem 3.45. *Let M be an even dimensional spin manifold and $\eth_F^2$ the spin Dirac operator twisted by a bundle $F \to M$. Then the heat kernel $e^{-t\eth_F^2}$ is an element of ${}^{\mathrm{G}}\Phi^{2n}$ with*

$$\begin{aligned} &{}^{\mathrm{G}}N(e^{-t\eth_F^2})(x,\zeta) \\ &= (4\pi)^{-n}\det\Big(\frac{\mathsf{R}(x)/2}{\sinh\mathsf{R}(x)/2}\Big)^{1/2}\exp\big(-\tfrac{1}{4}\langle\mathsf{R}(x)/2\coth(\mathsf{R}(x)/2)\zeta,\zeta\rangle\big)\exp\big(-K^F(x)\big). \end{aligned} \tag{3.63}$$

In particular, $\operatorname{ind}(\eth_F)$ *is given by the* **local index formula**

$$\operatorname{ind}(\eth_F) = \operatorname{Str} e^{-t\eth_F^2}|_{t=0} = (2\pi i)^{-n}\int_M\Big[\det\Big(\frac{\mathsf{R}/2}{\sinh\mathsf{R}/2}\Big)^{1/2}\operatorname{str}_F\exp(-K^F)\Big]_{2n}. \tag{3.64}$$

Remark. The formula (3.64) is usually attributed to Gilkey and Patodi, who originally proved it using slightly different methods.

Proof. The solution (3.62) can be put together fiber by fiber on $\mathsf{hf} \to M$ to define the solution (3.63) to the model operator $-\frac{1}{2}\zeta \cdot \partial_\zeta + H_\zeta + K^F \in \mathrm{Diff}^2_{\mathrm{fib}}\big(TM; \Lambda M \otimes \mathrm{End}(F)\big)$. We can choose $H_1 \in {}^{\mathrm{G}}\Phi^{2n}$ with ${}^{\mathrm{G}}N(H_1)$ equal to (3.63) and then $t(\partial_t + \eth_F^2)H_1 =: R_1 \in {}^{\mathrm{G}}\Phi^{2n-1}$. This same fundamental solution can be used to solve away ${}^{\mathrm{G}}N(R_1)$ over TM, so we can proceed iteratively as we did in §2.3.5, obtaining $H = H_1 + H_2 + \cdots$ such that $t(\partial_t + \eth_F^2)H \in {}^{\mathrm{G}}\Phi^{-\infty}$. This final error can be removed by convolution as in §2.3.7. As a result, we obtain the true fundamental solution $e^{-t\eth_F^2} \in {}^{\mathrm{G}}\Phi^{2n}$, whose normal operator agrees with that of H_1, i.e., is given by (3.63).

The equation (3.64) follows from Propositions 3.41 and 3.32; we are using the fact that the $2n$-form component of ${}^{\mathrm{G}}N(e^{-t\eth_F^2})$ can be written in terms of the volume element $e_1 \cdots e_{2n}$, and from Proposition 3.32, $\mathrm{str}(e_1 \cdots e_{2n}) = (-2i)^n$. □

The index formula (3.64) gives a prescribed *index density*, i.e., a top degree form whose total integral over M is the index of the operator. We will show in the next section that (3.64) has an interpretation in cohomology (which is usually what we refer to by "the Atiyah-Singer index formula"), but this analytical result is strictly stronger since it prescribes a de Rham representative for the class. One application of this is the following.

Suppose $\widetilde{M} \to M$ is a k-fold cover. Since it is a local diffeomorphism, there is a canonical lift of $\eth_F$ to a differential operator $\widetilde{\eth}_F$ on $\widetilde{M}$. The heat kernel construction is completely local for $t \to 0$, so it follows that the index density for $\widetilde{\eth}_F$ is the obvious lift of the index density for $\eth_F$ from M, and it then follows that

$$\mathrm{ind}(\widetilde{\eth}_F) = k\,\mathrm{ind}(\eth_F),$$

i.e., that the index is multiplicative with respect to finite covers. This fact does not follow from the cohomological formulation of the index theorem.

3.4.1 Non-spin manifolds

Before going into the cohomological formulation, let us address the requirement that M be spin. As we noted before, admitting a spin structure is a global property of a manifold; there is no obstruction locally. Since the index formula in Theorem 3.45 is completely local, it cannot detect whether or not M is spin, so it should hold as well for general Dirac operators on non-spin manifolds. Indeed, one way to see this is to choose spin structures locally, decompose a Dirac operator with respect to these choices, and then check that the index density does not depend on the local choices.

More directly, we can make the following observation. Suppose $E \to M$ is a Clifford module. If M is not spin, then we cannot generally write $E = \mathrm{S} \otimes F$ for some F. However, it *is* always true that

$$\mathrm{End}(E) \cong \mathbb{C}\ell(M) \otimes \mathrm{End}_{\mathbb{C}\ell}(E), \tag{3.65}$$

where the second factor denotes endomorphisms of E which commute with the Clifford action. In the case that $E = \mathrm{S} \otimes F$, then $\mathrm{End}_{\mathbb{C}\ell}(E) = \mathrm{End}(F)$ and $\mathbb{C}\ell(M) = \mathrm{End}(\mathrm{S})$. Given a Clifford connection ∇^E on E, we can write its curvature tensor as

$$K^E = R^{\mathrm{S}} + K^{E/\mathrm{S}} \in C^\infty\big(M; \Lambda^2 \otimes (\mathbb{C}\ell(M) \otimes \mathrm{End}_{\mathbb{C}\ell}(E))\big),$$

where $R^{\mathbb{S}}$ is the Riemannian curvature acting on $\mathbb{C}\ell(M)$ and $K^{E/\mathbb{S}} := K^E - R^{\mathbb{S}}$.

Exercise 3.6. Show that $K^{E/\mathbb{S}}$ is a section of $\Lambda^2 \otimes \mathrm{End}_{\mathbb{C}\ell}(E)$ by showing that it commutes with any clifford multiplication $c\ell(v)$, where v is a section of TM.

Definition 3.46. The term $K^{E/\mathbb{S}} \in C^\infty\big(M; \Lambda^2 \otimes \mathrm{End}_{\mathbb{C}\ell}(E)\big)$ is called the **twisting curvature** of ∇^E. In the case that $E = \mathbb{S} \otimes F$, $\nabla^E = \nabla \otimes 1 + 1 \otimes \nabla^F$, then $K^{E/\mathbb{S}}$ is the curvature of ∇^F.

Let w be a section of $\mathrm{End}_{\mathbb{C}}\,\ell(E)$. We define the **relative supertrace** of s by

$$\mathrm{str}_{E/\mathbb{S}}(w) := 2^{n/2}\,\mathrm{str}_E(\omega_{2n} w),$$

where $\omega_{2n} = i^n e_1 \cdots e_{2n} \in \mathbb{C}\ell(M)$ is the Clifford volume element. In the case that $E = \mathbb{S} \otimes F$, $w = s \otimes f$, then $\mathrm{str}_E(w) = \mathrm{str}_{\mathbb{C}\ell}(s)\,\mathrm{str}_F(f)$ (since $\mathrm{str}(\omega_{2n}) = 2^n$).

Since the construction of the heat kernel only involves the endomorphism bundle $\mathrm{END}(E)$, and the decomposition (3.65) is canonical, the proof of Theorem 3.45 goes through to prove the following generalization.

Theorem 3.47 (Local index theorem)**.** *Let $D \in \mathrm{Diff}^1(M; E)$ be a Dirac operator associated to a Clifford module $E \to M$ with Clifford connection ∇^E on any Riemannian manifold M of dimension $2n$. Assume $E = E^0 \oplus E^1$ is graded, so*

$$D = \begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix}$$

with $D_0 \in \mathrm{Diff}^1(M; E^0, E^1)$ and $D_1 = D_0^*$. Then

$$\mathrm{ind}(D_0) = \mathrm{Str}\, e^{-tD^2}|_{t=0} = (2\pi i)^{-n} \int_M \Big[\det\Big(\frac{\mathsf{R}/2}{\sinh \mathsf{R}/2}\Big)^{1/2} \mathrm{str}_{E/\mathbb{S}} \exp(-K^{E/\mathbb{S}})\Big]_{2n}.$$

3.4.2 A bit of Chern-Weil theory

Chern-Weil theory is essentially the "de Rham version" of characteristic classes. We briefly recall here how it goes. Let $E \to M$ be a vector bundle (possibly with extra structure, such as a unitary or orthogonal structure with respect to an inner product) and ∇ a connection on E. Throughout this section we will also denote the *exterior covariant derivative* by $\nabla : C^\infty(M; \Lambda^k \otimes E) \to C^\infty(M; \Lambda^{k+1} \otimes E)$, which we recall is the unique extension of ∇ to differential forms such that $\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|}\omega \wedge \nabla s$.

Denote the curvature of ∇ by $K \in C^\infty\big(M; \Lambda^2 \otimes \mathrm{End}(E)\big)$; in terms of the exterior covariant derivative, recall that $K = \nabla^2$, which leads to an obvious proof of **Bianchi's identity**

$$[\nabla, K] = 0.$$

Now let $f(z) \in \mathbb{C}[[z]]$ be a polynomial. The differential form $f(K) \in C^\infty\big(M; \Lambda M \otimes \mathrm{End}(E)\big)$ is well-defined (only finitely many terms are nonvanishing since $\Lambda^l M = \{0\}$ for sufficiently large

l), where the product is taken both as differential forms and as composition in $\mathrm{End}(E)$. By the noncommutativity of the latter, $K^2 \neq 0$ in general. Taking the trace then gives a well-defined total differential form:

$$\operatorname{tr} f(K) \in C^\infty(M; \Lambda M).$$

Remark. If G is the structure group of E (say $\mathrm{SO}(k)$ or $\mathrm{U}(k)$, for example), then K is a section of $\Lambda^2 \otimes \mathfrak{g}(E)$, and we may more generally consider $p(K)$ where $p \in \mathbb{C}[\mathfrak{g}^*]^G$ is any *invariant polynomial* (or power series). However it suffices for our purposes to restrict to those invariant polynomials/power series which are given by the trace of an ordinary polynomial/power series.

Proposition 3.48.

(i) *The total form* $\operatorname{tr} f(K)$ *is closed, i.e.,* $d \operatorname{tr} f(K) = 0$.

(ii) *If* ∇^t, $t \in [0,1]$ *is a one parameter family of connections, with curvature* K^t, *then* $\partial_t \operatorname{tr} f(K)$ *is exact.*

(iii) *In particular, the cohomology class* $[\operatorname{tr} f(K)] \in H^*(M; \mathbb{C})$ *is independent of the choice of connection.*

Proof. If $A \in C^\infty\big(M; \Lambda^k \operatorname{End}(E)\big)$ is any $\mathrm{End}(E)$-valued k-form, then a general formula says that $d \operatorname{tr} A = \operatorname{tr} \nabla(A) = \operatorname{tr}[\widetilde{\nabla}, A]_s$ where we use the *supercommutator* (3.38) with respect to the $\mathbb{Z}_2$ grading $(\Lambda^{\mathrm{even}} M \oplus \Lambda^{\mathrm{odd}} M) \otimes \operatorname{End}(E)$ This can be verified locally, after noting that the right hand side is independent of the connection used (since $\operatorname{tr}[\nabla - \nabla, A]_s = \operatorname{tr}[\nabla - \nabla', A] = 0$ as $\nabla - \nabla' \in C^\infty(M; \Lambda^1 \otimes \operatorname{End}(E)$ and tr vanishes on commutators); in particular you can use a trivial connection locally. Then (i) follows from $d \operatorname{tr} K^l = \operatorname{tr}[\nabla, K^l]_s = \operatorname{tr}[\nabla, \widetilde{\nabla}^{2l}] = 0$.

Suppose ∇^t is a one-parameter family of connections; in particular $\dot{\nabla}^t := \partial_t \nabla^t \in C^\infty\big(M; \Lambda^1 \otimes \operatorname{End}(E)\big)$ is an endomorphism-valued 1-form. From $K^t = (\nabla^t)^2$ we have $\dot{K}^t = [\widetilde{\nabla}^t, \dot{\nabla}^t]_s$, and

$$\partial_t \operatorname{tr} f(K^t) = \operatorname{tr} f'(K^t) \dot{K}^t = \operatorname{tr} f'(K^t) [\nabla^t, \dot{\nabla}^t]_s = \operatorname{tr}[\nabla^t, f'(K^t)\dot{\nabla}^t]_s = d \operatorname{tr}(f'(K^t)\dot{\nabla}^t),$$

where we have used the fact that $[\nabla^t, f'(K^t)]_s = 0$.

In particular, if ∇^0 and ∇^1 are two connections on E, then $\nabla^t := t\nabla^1 + (1-t)\nabla^0$ is a one-parameter family, and

$$\operatorname{tr} f(K^1) - \operatorname{tr} f(K^0) = \int_0^1 \partial_t \operatorname{tr} f(K^t)\, dt = d \int_0^1 \operatorname{tr}(f'(K^t)\dot{\nabla}^t)\, dt,$$

so $[\operatorname{tr} f(K^1)] = [\operatorname{tr} f(K^0)] \in H^*(M; \mathbb{C})$, proving (iii). □

Definition 3.49. The (total) cohomology class $f(E) := [\operatorname{tr} f(K)] \in H^*(M; \mathbb{C})$ is called the **characteristic class** of E associated to the power series $f \in \mathbb{C}[[z]]$. It is *functorial* in that if $\phi : N \to M$ is a smooth map of manifolds, then $f\big(\phi^*(E)\big) = \phi^* f(E) \in H^*(N; \mathbb{C})$.

Remark. The characteristic classes as we have defined them here can be shown to be equivalent (modulo torsion) to characteristic classes defined in algebraic topology, as cohomology classes in $H^*(BG;\mathbb{Z})$ where BG is the *classifying space* for the structure group of E. In fact it can be shown that the basic characteristic class $[\operatorname{tr} K] \in H^2(M; 2\pi i\mathbb{Z})$, i.e., is $2\pi i$ times an integral class (which is the first Chern class of E in this example). For this reason it is common to follow the convention of replacing K by $K/2\pi i$ in the definitions above in order to recover the integral characteristic classes of algebraic topology (again, modulo torsion).

Example 3.50.

(a) The **Chern character** is the characteristic class

$$\mathrm{Ch}(E) = \operatorname{tr}\exp(-K/2\pi i) \in H^{2*}(M;\mathbb{C})$$

associated to the power series $f(z) = e^{-z}$. It has the additivity and multiplicativity properties[16]

$$\mathrm{Ch}(E \oplus F) = \mathrm{Ch}(E) + \mathrm{Ch}(F), \quad \mathrm{Ch}(E \otimes F) = \mathrm{Ch}(E)\mathrm{Ch}(F)$$

hence the term “character”[17]. If $E^0 \oplus E^1$ is a $\mathbb{Z}_2$-graded bundle, then we can use a connection preserving the subbundles $E^{\pm}$, and in light of additivity, replacing the trace by the supertrace gives

$$\mathrm{Ch}_{\mathbb{Z}_2}(E) = \operatorname{str}\exp(K/2\pi i) = \mathrm{Ch}(E^0) - \mathrm{Ch}(E^1)$$

(b) The **A-hat class** of a real vector bundle $E \to M$ is the class

$$\widehat{A}(E) = \det{}^{1/2}\frac{K/4\pi i}{\sinh K/4\pi i} = \exp\operatorname{tr}\frac{1}{2}\log\frac{K/4\pi i}{\sinh K/4\pi i} \in H^{4*}(M;\mathbb{C}).$$

(We write the second formula to indicate how it may be associated to the trace of the power series $f(z) = \log\frac{\sqrt{z}/2}{\sinh\sqrt{z}/2}$.) The reason the total class only contains terms of degrees which are multiples of 4 is that $f(z)$ is even, hence has an expansion in z^2. The A-hat class is multiplicative:

$$\widehat{A}(E \oplus F) = \widehat{A}(E)\widehat{A}(F).$$

(c) The **Hirzebruch L-class** of a real vector bundle $E \to M$ is the class

$$L(E) = \det{}^{1/2}\frac{K/2\pi i}{\tanh K/2\pi i} \in H^{4*}(M;\mathbb{C})$$

The L-class is also multiplicative:

$$L(E \oplus F) = L(E)L(F).$$

[16] The second of these follows easily from the definition and the fundamental property of exponentials. Proving the first involves more theory than we shall develop here.

[17] In fact it defines a ring homomorphism from topological K-theory to cohomology, which was shown by Atiyah and Hirzebruch to be an isomorphism modulo torsion.

(d) The **Euler class** of an oriented real bundle $E \to M$ is

$$\chi(E) = \mathrm{Pf}(-K/2\pi i) = \det{}^{1/2}(-K/2\pi i) \in H^{2*}(M;\mathbb{C}).$$

Here we use the fact that E is oriented to choose a square root of the determinant known as the **Pfaffian**; if $A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$ is in $\mathfrak{so}(2)$, then $\mathrm{Pf}(A) = a$. Note that the sign of $\mathrm{Pf}(E)$ depends on the choice of orientation for E.

In the case that $E = TM$, we denote these classes simply by $\widehat{A}(M)$, $L(M)$, etc. The multiplicative classes are associated to so-called **genera**. For example, the **A-hat genus of** M is the (rational) number

$$\widehat{\mathbb{A}}(M) = \left\langle \widehat{A}(M), [M] \right\rangle = \int_M \widehat{A}(M) \in \mathbb{Q},$$

and the **L-genus of** M is

$$\mathbb{L}(M) = \langle L(M), [M] \rangle = \int_M L(M) \in \mathbb{Q}.$$

Remark. The genera are important in algebraic because they are cobordism invariant, and from the multiplicativity properties they define homomorphisms from cobordism rings to $\mathbb{Q}$.

As a corollary, we obtain the **cohomological form** of the index theorem:

Theorem 3.51 (Atiyah-Singer)**.** *Let M be a manifold of dimension $2n$, and $D \in \mathrm{Diff}^1(M;E)$ a Dirac operator associated to the graded Clifford module $E = E^0 \oplus E^1$. Then*

$$\mathrm{ind}(D_0) = \int_M \widehat{A}(M)\mathrm{Ch}_{\mathbb{Z}_2}(E/\mathbb{S}).$$

Proof. The formula follows immediately from the definitions above, where we have absorbed the constant $(2\pi i)^{-n}$ into the degree $2n$ portion of the characteristic class (which is the only portion contributing to the integral) to compensate for the replacement of curvature forms R and $K^{E/\mathbb{S}}$ by $\mathsf{R}/2\pi i$ and $K^{E/\mathbb{S}}/2\pi i$, respectively. □

3.4.3 Applications

The most direct application of the index formula is to the case of the spin Dirac operator $\eth \in \mathrm{Diff}^1(M;\mathbb{S})$ on a spin manifold M. In this case there is no twisting, so we obtain

Theorem 3.52. *The index of $\eth$ is the A-hat genus*

$$\mathrm{ind}(\eth^+) = \widehat{\mathbb{A}}(M).$$

In particular, the A-hat genus of a spin manifold must be an integer.

Applying Theorem 3.51 to other Dirac operators involves computing the Chern character of the twisting curvature. Since we are out of time, we will not go into details, but let us mention some of the highlights.

Let $D = (d + d^*) \in \mathrm{Diff}^1(M; \Lambda M)$, with the grading $\Lambda M = \Lambda^{\mathrm{even}} M \oplus \Lambda^{\mathrm{odd}} M$. Then

$$\mathrm{ind}(D_0) = \sum_{i=0}^{2n} (-1)^i \dim H^i(M; \mathbb{R}) = \mathrm{Eul}(M)$$

is the Euler characteristic of M. On the other hand, it can be shown that $\mathrm{Ch}_{\mathbb{Z}_2}\big((\Lambda^{\mathrm{even}} M \oplus \Lambda^{\mathrm{odd}} M)/\mathrm{S}\big) = i^{n/2} \chi(TM) \widehat{A}(TM)^{-1}$. From this we conclude.

Theorem 3.53 (Chern-Gauss-Bonet)**.** *Let M be a manifold of dimension $2n$. Then*

$$\mathrm{Eul}(M) = \mathrm{ind}(D_0) = \int_M \mathrm{Pf}(-R/2\pi).$$

In particular, if $\dim(M) = 2$, then

$$\mathrm{Eul}(M) = \frac{1}{2\pi} \int_M \kappa.$$

There is another grading on ΛM which follows from the identification $\Lambda M \cong \mathbb{C}\ell(M) = \mathbb{C}\ell^+(M) \oplus \mathbb{C}\ell^-(M)$, where we split into ± 1 eigenspaces of the Clifford volume element $\omega_{2n} = i^n e_1 \cdots e_{2n}$. On ΛM this is equivalent to the splitting into ± 1 eigenspaces of the Hodge star operator. There is a natural pairing

$$\begin{aligned} &H^j(M; \mathbb{R}) \times H^{2n-j}(M; \mathbb{R}) \to \mathbb{R}, \\ &\qquad ([\alpha], [\beta]) \mapsto \langle [\alpha] \wedge [\beta], [M] \rangle = \int_M \alpha \wedge \beta = \int_M \langle \alpha, \star\beta \rangle \, \mathrm{dVol} \end{aligned}$$

Which in particular gives a quadratic form on $H^n(M; \mathbb{R})$. If n is even (i.e., $\dim(M)$ is a multiple of 4), then this quadratic form is symmetric and we define the **signature of** M to be the signature of this quadratic form, denoted by $\sigma(M)$. It is not difficult to show that $\mathrm{ind}\big((d + d^*)_+\big) = \sigma(M)$ in this case, and computing the relative Chern character gives $\mathrm{Ch}_{\mathbb{Z}_2}\big((\Lambda^+ M \oplus \Lambda^- M)/\mathrm{S}\big) = L(M) \widehat{A}(M)^{-1}$.

Theorem 3.54 (Hirzebruch)**.** *Let M have dimension $m = 4n$. Then the signature of M coincides with the L-genus:*

$$\sigma(M) = \mathrm{ind}\big((d + d^*)_+\big) = \int_M L(M) = \mathbb{L}(M).$$

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