Representation theory for groups of unitary operators and an application in acoustic obstacle scattering

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Introduction

We consider the Cauchy problem for the wave equation

$$u_{tt} - \Delta u = 0,$$
$$u(x,0) = f_1(x), \qquad u_t(x,0) = f_2(x),$$

in $\mathbb{R}^n$ with $n \geq 3$ odd, with complex-valued initial data f_1, f_2. It is well known that under very general conditions on the initial data there exists a unique solution to this problem (which can be computed using the Fourier transform), called the free space solution. Suppose we introduce an *obstacle* into the space, that is, a bounded domain with smooth boundary, on which the solution to the wave equation must vanish. The aim of *scattering theory* is to study how the obstacle affects the solutions, compared to the free space solution. A prototypical application of this situation is *acoustic obstacle scattering*, the scattering of sound waves (which are governed by the wave equation) by an obstacle in three-dimensional space.

Now, in many physical applications, the obstacle is not directly accessible, and the only information available is how waves at different frequencies are scattered, by measuring these data far from the obstacle. Then one can try to gain information about the obstacle using these data. An important question is whether the measured data uniquely determine the obstacle. This is the *inverse scattering problem*, which has been thoroughly studied by mathematicians and physicists, due to its many applications. The tool employed in this study is the *scattering operator*, which relates the solution at a time sufficiently long before the wave hits the object to a time sufficiently long after; so the scattering operator encodes the measured data, and the goal of inverse scattering theory is to recover information about the obstacle from properties of the scattering operator.

In this paper, we want to present a construction of the scattering matrix for the acoustic scattering problem described above, using a group of unitary operators, as done in [1]. We can associate a group of operators $\{U(t), t \in \mathbb{R}\}$ to the wave equation, by letting $U(t)$ map initial data to the corresponding solution at time t. Since solutions to the wave equation conserve energy, these operators are unitary in the appropriate norm. In [1], this group is employed in the study of the scattering operator. The construction relies on representation theory for such a group, so in Section 1, we start by considering a general group of unitary operators and show the existence of two different representations, the *translation* and *spectral representations*. In Section 2, we study the wave equation in free space. We define the group $\{U_o(t)\}$ of unitary operators associated with this problem and explicitly construct its spectral and translation representations. In Section 3, we introduce an obstacle into space, at the boundary of which the solution of the wave equation must vanish. We construct the associated group $\{U(t)\}$ for this problem and apply the results from Section 1 to obtain representations for this group. Finally, we use these results together with those for the free space case obtained in Section 2 to define the scattering operator for obstacle scattering.

1 Preliminaries: Representations of a group $\{U(t)\}$ of unitary operators

We consider the strongly continuous group $\{U(t)\}$ of unitary operators acting on a Hilbert space H. We first introduce the notion of an *outgoing subspace* D_+ for $\{U(t)\}$.

Definition 1.1. A closed subspace D_+ of H is called *outgoing subspace for the group* $\{U(t)\}$, if it satisfies the following conditions:

(i) $U(t)D_+ \subset D_+$ for $t > 0$;

(ii) $\bigcap U(t)D_+ = \{0\}$;

(iii) $\overline{\bigcup U(t)D_+} = H$.

A space D_- is called *incoming subspace for* $\{U(t)\}$, if (i') $U(-t)D_- \subset D_-$ for $t > 0$, and (ii) and (iii) hold for D_-.

Our goal is to construct an *outgoing translation representation*, that is, a representation of $U(t)$ as right-translation by t, on $L^2(-\infty, \infty; N)$, the space of square-integrable functions on $\mathbb{R}$ with values in an auxiliary Hilbert space N, such that D_+ corresponds to $L^2(0, \infty; N)$.

In a first step, we replace the continuously parameterized group $\{U(t)\}$ by a single unitary operator and its powers, by means of the Cayley transform of the infinitesimal generator of the group $\{U(t)\}$.

By Stone's Theorem [3, Sec.35.1, Thm. 1], the generator A of $\{U(t)\}$ is skew-selfadjoint, thus the spectrum of A consists of purely imaginary numbers only, and in particular the values $\lambda = \pm 1$ are in the resolvent set of A. Therefore, the Cayley transform of A,

$$V = (I + A)(I - A)^{-1} \tag{1.1}$$

is a well-defined map from H onto itself. Now, if $x \in D(V) = \mathit{Range}(V) = H$, then there is $y \in D(A)$ such that $x = (I - A)y$, and, applying V, $Vx = (I + A)y$. Thus,

$$|x|^2 = (x, x) = (y - Ay, y - Ay) = |y|^2 - (y, Ay) - (Ay, y) + |Ay|^2 = |y|^2 + |Ay|^2 = |Vx|^2,$$

having used the skew-symmetry of A, and this shows that V is a unitary operator. We also note that the Cayley transform of $-A$ is V^{-1}. We define the notion of outgoing subspace for V analogously as for $\{U(t)\}$:

Definition 1.2. A closed subspace D_+ of H is called *outgoing subspace for* V if

(i) $VD_+ \subset D_+$;

(ii) $\bigcap V^k D_+ = \{0\}$;

(iii) $\overline{\bigcup V^k D_+} = H$.

A closed subspace D_- is called *incoming subspace for* V if (i') $V^{-1}D_- \subset D_-$, and (ii), (iii) hold for D_-.

In the following, we will construct translation and spectral representations for V. First, we proceed by establishing a series of lemmas that will allow us to eventually recover a representation for $\{U(t)\}$ from that for V.

Lemma 1.3. If D is a subspace of H such that $U(t)D \subset D$ for all $t > 0$, then $VD \subset D$, and vice versa.

Proof. Denote by $R(\lambda, A) = (\lambda I - A)^{-1}$ the resolvent of A, and note that we can express $V = R(1, A) + A\,R(1, A)$, whence, adding and subtracting $R(1, A)$, we obtain

$$V = 2R(1, A) - I. \tag{1.2}$$

Now the resolvent of A has the following Laplace transform representation [3, Sec.34.1 Thm. 4],

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} U(t) dt,$$

which in turn provides the representation

$$Vx = 2\int_0^\infty e^{-t} U(t)x\, dt - x.$$

Thus, if for $x \in D$, we have $U(t)x \in D$ for all $t > 0$, then the above identity shows $Vx \in D$.

To show the converse, we start by showing that if $VD \subset D$, then $R(\lambda, A)D \subset D$ for all $\lambda > 0$: Since the resolvent is an analytic function in λ on the resolvent set (e.g. [3, Sec.17.1, Thm. 3]), it has a power series expansion

$$R(\lambda, A) = \sum_{m=0}^\infty (\lambda_o - \lambda)^m R(\lambda_o, A)^{m+1}, \quad \text{for} \quad |\lambda_o - \lambda| \|R(\lambda_o, A)\| < 1.$$

Now the Hille-Yosida theorem [3, Sec. 34.2 Thm. 7] states that for $\lambda_o > 0$, $\|R(\lambda_o, A)\| \le 1/\lambda_o$, so the series expansion holds for $|\lambda_o - \lambda| < \lambda_o$. This expansion shows that if $R(\lambda_o, A)D \subset D$, then also $R(\lambda, A)D \subset D$ for all λ such that $|\lambda - \lambda_o| < \lambda_o$.

Thus, if $VD \subset D$, then (1.2) shows that $R(1, A)D \subset D$, and now we can inductively use the above argument to see that indeed $R(\lambda, A)D \subset D$ for all $\lambda > 0$. Therefore, if $x \in D$, and $y \in D^\perp$, the orthogonal complement of D, then

$$0 = (R(\lambda, A)x, y) = \int_0^\infty e^{-\lambda t}(U(t)x, y)dt$$

for all $\lambda > 0$. Now uniqueness of the Laplace transform implies $(U(t)x, y) = 0$, whence $U(t)D \subset D$ for all $t > 0$. ☐

Corollary 1.4. We have $U(t)D \subset D$ for all $t \in \mathbb{R}$ if and only if $V^k D \subset D$ for all $k \in \mathbb{Z}$. In either case, $U(t)D = D = V^k D$ for all t and k.

Proof. We apply the lemma to both $\{U(t), t \ge 0\}$ and $\{U(-t), t \ge 0\}$, to obtain

$$U(t)D \subset D \ \ \forall t > 0 \ \ \Leftrightarrow VD \subset D,$$
$$U(-t)D \subset D \ \ \forall t > 0 \Leftrightarrow V^{-1}D \subset D.$$

That $U(t)D = D = V^k D$ then follows from the group property of $\{U(t)\}$ and $\{V^k\}$, since $M = U(-t)U(t)M \subset U(-t)M \subset M$, and similarly for V. ☐

Lemma 1.5. A closed subspace D is outgoing (or incoming) for $\{U(t)\}$ if and only if it is outgoing (or incoming) for V.

Proof. The equivalence of (i) for V and $\{U(t)\}$ is just Lemma 1.3. For (ii), we define

$$P = \bigcap_{t\in\mathbb{R}} U(t)D, \qquad P' = \bigcap_{k\in\mathbb{Z}} V^k D.$$

We want to show that $P = P'$. Note first that $U(t_o)P = \bigcap U(t + t_o)D = P$ for all $t_o \in \mathbb{R}$, and in the same way $V^k P' = P'$ for all $k \in \mathbb{Z}$. Thus, by Corollary 1.4, also $U(t)P' = P'$ for all t and $V^k P = P$ for all k. Since both P and P' are subsets of D, we get the sequence of inclusions

$$P = \bigcap V^k P \subset \bigcap V^k D = P' = \bigcap U(t)P' \subset \bigcap U(t)D = P,$$

thus $P = P'$, whence we infer the equivalence of the two conditions (ii). For (iii), we proceed analogously and set

$$M = \overline{\bigcup U(t)D}, \qquad M' = \overline{\bigcup V^k D}.$$

We need to show $M = M'$. Note that both M and M' contain D; furthermore, we have again $U(t)M = M$ for all t and $V^k M = M$ for all k, so we can employ Corollary 1.4 to get $U(t)M' = M'$ and $V^k M = M$, and the sequence of inclusions (using that M, M' are closed)

$$M = \bigcup V^k M \supset \overline{\bigcup V^k D} = M' = \bigcup U(t)M' \supset \overline{\bigcup U(t)D} = M,$$

and this establishes the equivalence of the two conditions (iii). □

1.1 Translation and spectral representations for V

We start with the construction of an *outgoing translation representation* for V, presented in the following theorem, where for a Hilbert space N, we denote by $\ell^2(-\infty, \infty; N)$ the space of all sequences $\{y_k,\ k \in \mathbb{Z}\}$ such that $y_k \in N$ for all k, and $\sum \|y_k\|_N^2 < \infty$.

Theorem 1.6. If V is a unitary operator on the Hilbert space H, and $D \subset H$ is an outgoing subspace for V, then H can be represented isometrically as $\ell^2(-\infty, \infty; N)$ for some auxiliary Hilbert space N, in such a way that V goes into the right shift operator and D maps onto $\ell^2(0, \infty; N)$. This representation is unique up to an isomorphism of N.

Proof. By assumption on D, we have $VD \subset D$. We take N to be the orthogonal complement of VD in D, which we write as

$$N = D \ominus VD.$$

We are going to prove

$$D = \bigoplus_{k\geq 0} V^k N, \quad \text{and} \tag{1.3}$$

$$H = \bigoplus_{k\in\mathbb{Z}} V^k N. \tag{1.4}$$

Set $M = \bigoplus_{k\geq 0} V^k N$. Since V is unitary, we have

$$V^k N = V^k D \ominus V^{k+1} D.$$

Furthermore, property (i) of D implies $V^{k+1}D \subset V^kD$ for all k. So if $k < l$, we find

$$V^kN \perp V^{k+1}D \supset V^lD \supset V^lN,$$

and we see that the V^kN are mutually orthogonal subspaces of D, so that $M \subset D$.

If we assume M is a proper subspace of D, then there is a nonzero $x \in D \ominus M$. So $x \perp N \subset M$, therefore $x \in VD$. We also have $x \perp VN$, and thus $x \in V^2D = VD \ominus VN$. Continuing this argument, we find that $x \in \bigcap V^kD$, which yields a contradiction since by property (ii) of D, this would imply $x = 0$. So we have established (1.3). Using this identity, we get

$$V^kD = \bigoplus_{j\geq k} V^jN \subset \bigoplus_{j\in\mathbb{Z}} V^jN,$$

for all k, thus $\bigcup V^kD \subset \bigoplus V^jN$, and since by property (iii) of D, $\bigcup V^kD$ is dense in H and the right-hand space is a closed subspace of H, (1.4) follows.

To show that for this space N, H is indeed isometrically isomorphic to $\ell^2(-\infty,\infty;N)$, we proceed to construct the isomorphism. We note that by (1.4), each $x \in H$ can be uniquely decomposed as

$$x = \sum_{k\in\mathbb{Z}} V^k y_k, \quad y_k \in N. \tag{1.5}$$

By orthogonality and the fact that V is unitary, we then get

$$\|x\|_H^2 = \sum_{k\in\mathbb{Z}} \|V^k y_k\|_H^2 = \sum_{k\in\mathbb{Z}} \|y_k\|_N^2,$$

where $\|.\|_N$ denotes the induced norm on N. On the other hand, each sequence $\{y_k\} \subset N$ with $\sum \|y_k\|_N^2 < \infty$ defines an element of H by (1.5). So the mapping

$$x \mapsto \{y_k\}$$

is an isometry of H onto $\ell^2(-\infty,\infty;N)$, and by (1.3), D is mapped onto $\ell^2(0,\infty;N)$. Finally, we check that

$$Vx = \sum V^{k+1} y_k \mapsto \{y_{k-1}\},$$

so V becomes the right-shift operator under this mapping. □

By means of the Fourier transform, we can obtain the *outgoing spectral representation* for V, that is, a representation under which V goes to a multiplication operator. We recall the *Hardy space* $H_2(N)$ of functions on the unit circle whose k^{th} Fourier coefficients vanish for all $k < 0$. Another characterization of this space is that each $f \in H_2(N)$ is the boundary value (in the L^2 sense) of an analytic function on the unit disk whose square integral over circles centered at the origin is uniformly bounded (see [4, Sec. 17] for a formal definition and properties).

Corollary 1.7. If D is outgoing with respect to the unitary operator V, then H can be represented isometrically as $L^2(0,2\pi;N)$ for some auxiliary Hilbert space N so that V goes into multiplication by $e^{i\theta}$ and D is mapped onto $H_2(N)$, the Hardy space of functions on the unit circle whose k^{th} Fourier coefficients vanish for all $k < 0$. This representation is unique up to an isomorphism of N.

Proof. The map

$$\{y_k\} \mapsto f(\theta) = \sum_{k\in\mathbb{Z}} y_k e^{ik\theta}$$

from $\ell^2(-\infty,\infty;N)$ to $L^2(0,2\pi;N)$ with inverse

$$f(\theta) \mapsto \{y_k\}, \quad y_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-i\theta}d\theta \tag{1.6}$$

is an isomorphism of $\ell^2(-\infty,\infty;N)$ onto $L^2(0,2\pi;N)$. It is isometric since

$$\|f\|_{L^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} \|f(\theta)\|_N^2 d\theta = \sum_{k\in\mathbb{Z}} \|y_k\|_N^2 = \|\{y_k\}\|_{\ell^2}^2.$$

Note that for the right-shift $\{y_{k-1}\}$ of the sequence $\{y_k\}$,

$$\{y_{k-1}\} \mapsto \sum_{k\in\mathbb{Z}} y_{k-1}e^{ik\theta} = e^{i\theta}\sum_{k\in\mathbb{Z}} y_k e^{ik\theta},$$

so the right-shift operator corresponds to multiplication by $e^{i\theta}$; also, $\ell^2(0,\infty;N)$ clearly gets mapped onto $H_2(N)$. Combining this with the isomorphism from Theorem 1.6, we get the desired isomorphism for H. The uniqueness statement follows by using the inverse (1.6) and the corresponding result for the translation representation. □

Remark. For an incoming subspace D_- for V, one can obtain an *incoming* translation representation, such that D_- is mapped onto $\ell^2(-\infty,-1;N')$, and in the corresponding spectral representation, D_- is mapped onto $\bar{H}_2(N')$, the conjugate Hardy space of functions whose k^{th} Fourier coefficients vanish for $k \geq 0$. The auxiliary Hilbert spaces N and N' are in fact unitarily equivalent, as follows from the following, more general, theorem. For the proof, see [1, Sec.II, Thm. 1.2].

Theorem 1.8. Let V be a unitary operator. If there are two translation representations, say $\ell^2(-\infty,\infty;N)$ and $\ell^2(-\infty,\infty;N')$, then N and N' are unitarily equivalent. □

1.2 Spectral and translation representations for $\{U(t)\}$

We start by using the $L^2(0,2\pi;N)$ spectral representation for V to obtain a spectral representation for the group $\{U(t)\}$. The spectral representation for the group is constructed using its infinitesimal generator A: we find an isometry from H onto $L^2(-\infty,\infty;N)$, such that A goes into multiplication by $i\sigma$. Consequently, the operators $U(t)$ will be represented by the multiplication operators $e^{i\sigma t}$.

We employ the following transform mapping the unit disk in $\mathbb{C}$ onto the upper half-plane:

$$z = i\frac{1-w}{1+w}, \quad \text{with inverse} \quad w = \frac{1+iz}{1-iz}. \tag{1.7}$$

In particular, the circle gets mapped onto the real line via

$$e^{i\theta} \mapsto \sigma = i\frac{1-e^{i\theta}}{1+e^{i\theta}}.$$

With this notation, we define a map from $L^2(0,2\pi;N)$ to $L^2(-\infty,\infty;N)$ by

$$g(e^{i\theta}) \mapsto f(\sigma) = \frac{1}{\sqrt{\pi}}(1-i\sigma)^{-1}g\Big(\frac{1+i\sigma}{1-i\sigma}\Big). \tag{1.8}$$

This map is an isometry, since a change of variables shows that

$$\frac{1}{2\pi}\int_0^{2\pi} \|g(e^{i\theta})\|_N^2 d\theta = \int_{-\infty}^{\infty} \|f(\sigma)\|_N^2 d\sigma.$$

Furthermore, note that

$$e^{i\theta} g(e^{i\theta}) \mapsto \frac{1+i\sigma}{1-i\sigma} f(\sigma),$$

that is, multiplication by $e^{i\theta}$ goes into multiplication by $\frac{1+i\sigma}{1-i\sigma}$. We denote by M_+ and M_- the images of $H_2(N)$ and $\bar{H}_2(N)$, respectively, under this map. The characterization of elements of the Hardy spaces as boundary values of analytic functions on the disk and the map (1.7) imply that maps in M_+ are boundary values in the L^2 sense of functions analytic in the upper half plane whose square integral along the lines $\text{Im}z = \text{const} > 0$ is uniformly bounded; the analogous statement holds for M_- and functions analytic in the lower half-plane. By the Paley-Wiener theorem [3, Sec.37.7, Thm. 11], $f \in M_+$ is the Fourier transform[1], denoted by $\mathcal{F}$, of a square integrable function on $(0, \infty)$, and $f \in M_-$ is that of a square integrable function on $(-\infty, 0)$, and the converse holds as well. Thus,

$$M_+ = \mathcal{F} L^2(0, \infty; N), \qquad M_- = \mathcal{F} L^2(-\infty, 0; N).$$

We summarize these findings in the following lemma.

Lemma 1.9. The map (1.8) is an isomorphism of $L^2(0, 2\pi; N)$ onto $L^2(-\infty, \infty; N)$, mapping $H_2(N)$ onto M_+. The multiplicative operator $e^{i\theta}$ goes into the multiplicative operator $\frac{1+i\sigma}{1-i\sigma}$. □

We can now piece the foregoing together to prove the existence of an *outgoing spectral representation* for $\{U(t)\}$:

Theorem 1.10. If D is an outgoing subspace with respect to the strongly continuous group $\{U(t)\}$ of unitary operators, then H can be represented isometrically as $L^2(-\infty, \infty; N)$ for some auxiliary Hilbert space N, so that $U(t)$ goes into multiplication by $e^{i\sigma t}$ and D is mapped onto M_+. This representation is unique up to an isomorphism of N.

Proof. By Lemma 1.5, D is an outgoing subspace for V, the Cayley transform of the infinitesimal generator of the group. Thus, Theorem 1.6 and Corollary 1.7 provide a spectral representation for V in $L^2(0, 2\pi; N)$, such that D is mapped onto $H_2(N)$, and V corresponds to multiplication by $e^{i\theta}$. The transformation given in Lemma 1.9 maps $L^2(0, 2\pi; N)$ onto $L^2(-\infty, \infty; N)$, $H_2(N)$ onto M_+, and the multiplicative operator $e^{i\theta}$ into the multiplicative operator $\frac{1+i\sigma}{1-i\sigma}$. Note that the latter is the Cayley transform of multiplication by σ on $L^2(-\infty, \infty; N)$ (recall the definition (1.1) of the Cayley transform of an operator). This operator, multiplication by σ, is the infinitesimal generator of the group of multiplicative operators $\{e^{i\sigma t}\}$ on $L^2(-\infty, \infty; N)$. Now the Cayley transform uniquely determines the generator, and the generator in turn uniquely determines the group, so since the Cayley transform of the generator of $\{U(t)\}$ is mapped to that of $\{e^{i\sigma t}\}$, we conclude that $U(t)$ is mapped to $e^{i\sigma t}$ in this representation.

The uniqueness up to isomorphism follows from the corresponding result for the representation for V: If there are two distinct outgoing spectral representations for $\{U(t)\}$, we obtain two distinct outgoing translation representations for V by reversing the above procedure. Now by Theorem 1.6, these must be equal up to an isomorphism of N, and going back to the spectral representations for $\{U(t)\}$, these can differ only by an isomorphism. □

[1] we define the Fourier transform by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{\xi \cdot x} dx,$$

We finally obtain an outgoing translation representation for $\{U(t)\}$:

Corollary 1.11. If D is an outgoing subspace with respect to the group $\{U(t)\}$ of unitary operators, then H can be represented isometrically as $L^2(-\infty, \infty; N)$ for some auxiliary Hilbert space N so that $U(t)$ goes into right translation by t units and D is mapped onto $L^2(0, \infty; N)$. This representation is unique up to an isomorphism of N.

Proof. We apply the inverse Fourier transform which is unitary on $L^2(-\infty, \infty; N)$; by the Paley-Wiener theorem, the subspace M_+ gets mapped onto $L^2(0, \infty; N)$, and multiplication by $e^{i\sigma t}$ turns into a right shift. □

Note that again, one can analogously find incoming representations for $\{U(t)\}$: In the incoming spectral representation, D_- is mapped onto M_-, and in the incoming translation representation, D_- is mapped onto $L^2(-\infty, 0; N')$. As before for V, it can be shown that the auxiliary Hilbert spaces N' and N for the incoming and outgoing representations, respectively, are unitarily equivalent and may thus be identified.

2 The wave equation in free space

Now we turn to the study of the wave equation. In this section, we consider the problem in free space; we will explicitly construct the spectral and translation representations in this case. These will be used later on to define the scattering operator for the case of scattering by an obstacle in $\mathbb{R}^n$. We restrict our analysis to odd dimensions $n \geq 3$, and consider

$$u_{tt} - \Delta u = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{2.1}$$

$$u(x,0) = f_1(x), \quad u_t(x,0) = f_2(x), \qquad x \in \mathbb{R}^n, \tag{2.2}$$

with complex valued initial data $f = \{f_1, f_2\}$. For $f = \{f_1, f_2\} \in C_c^\infty(\mathbb{R}^n)^2$, we define the energy norm of f by

$$\|f\|_E^2 = \frac{1}{2} \int |\nabla f_1|^2 + |f_2|^2 dx,$$

and we denote by H_o the completion of $C_c^\infty(\mathbb{R}^n)^2$ with respect to this norm. H_o is a Hilbert space, containing the Cauchy data for the wave equation. It is useful to note that $f \in H_o$ means that $f_2 \in L^2(\mathbb{R}^n)$, and $f_1 \in L^2_{loc}(\mathbb{R}^n)$; for the latter property, see [1, Sec. IV.1].

Concerning solvability and properties of solutions of (2.1)-(2.2), we have the following classical results, see e.g. [2, Sec. 2.4].

Theorem 2.1. Given data $f = \{f_1, f_2\} \in C_c^\infty(\mathbb{R}^n)^2$, the initial value problem (2.1)-(2.2) has a unique solution u in $C^\infty(\mathbb{R}^n)$ with constant energy in t, that is

$$\frac{1}{2} \int |\nabla u(x,t)|^2 + |u_t(x,t)|^2 dx = \|f\|_E^2 \quad \text{for all } t \in \mathbb{R}.$$

Furthermore, we have *Huygens' Principle*: for odd $n \geq 3$, initial data at a point $x_o \in \mathbb{R}^n$ affects the solution only on the cone $\{|x - x_o| = t\}$.

The wave equation also allows for much less regular solutions (see [1, Sec.IV, Thm. 1.4]): Given a pair $\{f_1, f_2\}$ of distributions, there is a unique distribution solution $u(t)$ to (2.1)-(2.2) in the following sense: for any $\phi \in C_c^\infty(\mathbb{R}^n)$ the distributional pairing $(u(t),\phi)$ is a smooth function in t, with

$$(u(0),\phi) = (f_1,\phi), \qquad \frac{d}{dt}(u(t),\phi)\big|_{t=0} = (f_2,\phi).$$

We now construct a group of unitary operators as follows: For every t, we now define the operator $U_o(t)$ for compactly supported smooth initial data by

$$U_o(t)\{f_1,f_2\} = \{u(.,t),u_t(.,t)\},$$

i.e., $U_o(t)$ maps initial data to the corresponding solution of the wave equation at time t.

By Theorem 2.1, $U_o(t)$ maps C_c^∞ data into C_c^∞ data and forms a one-parameter group, which furthermore conserves energy. Hence it can be extended continuously to all of H_o to give a one-parameter group of unitary operators. Now by Stone's Theorem, this group has a skew-selfadjoint infinitesimal generator. Denote the infinitesimal generator of $\{U_o(t)\}$ by A_o. We take $f = \{f_1,f_2\} \in C_c^\infty(\mathbb{R}^n)^2$ to compute

$$A_o f = \lim_{t\to 0}\frac{U_o(t)f - f}{t} = \begin{pmatrix} u_t(x,0) \\ u_{tt}(x,0)\end{pmatrix} = \begin{pmatrix} u_t(x,0) \\ \Delta u(x,0)\end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0\end{pmatrix} f, \tag{2.3}$$

so on C_c^∞ data, $A_o = \begin{pmatrix} 0 & I \\ \Delta & 0\end{pmatrix}$. It can be shown that A_o on H_o is in fact the closure of this operator originally defined on compactly supported smooth functions. A proof of this fact using the spectral representation for A_o (which we construct below) is given in [1].

We proceed by defining the *outgoing* subspace $D_+ \subset H_o$ for $\{U_o(t)\}$.

Definition 2.2. Let $D_+ \subset H_o$ be the set of data for which the corresponding solution u of the wave equation (2.1)-(2.2) vanishes in the forward cone $\{|x| < t\}$; we further define the *incoming* subspace $D_- \subset H_o$ as the set of data for which the solution vanishes in the backward cone $\{|x| < -t\}$.

We call $f \in D_+$ outgoing data, and $f \in D_-$ incoming data. Furthermore, we call data f *eventually outgoing*, if there is some r such that $U_o(r)f$ is outgoing, i.e., the corresponding solution of the wave equation is zero on $\{|x| < t - r\}$.

The following result states that the spaces defined above are indeed outgoing and incoming according to Definition 1.1. A proof is given in [1, Sec.IV.2].

Proposition 2.3. D_+ is an outgoing subspace according to Definition 1.1 for $\{U_o(t)\}$. □

2.1 Spectral and translation representations for $\{U_o(t)\}$

We start by finding a unitary *spectral representation* of H_o for the group $\{U_o(t)\}$. More precisely, for $f \in H_o$, the spectral representation should be a square integrable function on the spectrum of A_o, given by the scalar product of f with eigenfunctions of A_o[2]:

$$\tilde{f}(\sigma) = (f,\phi_\sigma)_E. \tag{2.4}$$

[2]recall that the inner product on H_o is the energy scalar product

$$(f,g)_E = \frac{1}{2}\int \nabla f_1 \cdot \nabla \overline{g_1} + f_2\overline{g_2}dx.$$

where σ is an eigenvalue and ϕ_σ a corresponding eigenfunction.

Since A_o is skew-symmetric, it has purely imaginary eigenvalues $i\sigma$ with $\sigma \in \mathbb{R}$, with eigenfunctions denoted by ϕ_σ solving the eigenvalue equation

$$A_o\phi_\sigma = \begin{pmatrix} \phi_{\sigma,2} \\ \Delta\phi_{\sigma,1} \end{pmatrix} = i\sigma \begin{pmatrix} \phi_{\sigma,1} \\ \phi_{\sigma,2} \end{pmatrix}.$$

Note that $A_o = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ doesn't have any eigenfunctions in $(L^2(\mathbb{R}^n))^2$, but it has the following bounded eigenfunctions,

$$\phi_{\sigma,\omega}(x) = exp(-i\sigma\omega \cdot x) \begin{pmatrix} 1 \\ i\sigma \end{pmatrix}, \tag{2.5}$$

where $\omega \in \mathbb{S}^{n-1}$ is a unit vector. Since for every σ, ω can vary over the whole circle, there is an infinite number of bounded eigenfunctions for each eigenvalue, and the spectral representation will depend on ω as well, so we obtain a function on $\mathbb{R} \times \mathbb{S}^{n-1}$. Equivalently, to connect to the construction presented in the previous section, we can view $\tilde{f}$ as a function of σ with values in $N = L^2(\mathbb{S}^{n-1})$. The scalar product of spectral representations of f, $g \in H_o$, which we denote by square brackets, then is

$$[\tilde{f}, \tilde{g}] = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \tilde{f}(s,\omega)\overline{\tilde{g}(s,\omega)}\, ds\, d\omega.$$

In order for the spectral representation to be unitary, in view of (2.4) the eigenfunctions need to be weighted suitably. This is done in the following theorem.

Theorem 2.4. A unitary spectral representation of H_o for $\{U_o(t)\}$ is given by

$$\tilde{f}(\sigma,\omega) = \frac{(-i\sigma)^{(n-3)/2}}{(2\pi)^{n/2}}(f, \phi_{\sigma,\omega})_E \tag{2.6}$$

for $f \in H_o$, where $\phi_{\sigma,\omega}$ is the eigenfunction (2.5) of the infinitesimal generator A_o of the group.

Proof. Let $f \in \mathcal{S}^2$, $\mathcal{S}$ being the Schwartz space of smooth functions on $\mathbb{R}^n$ with rapidly decreasing derivatives. We first show that in this case $\tilde{f}$ defined as (2.6) belongs to $\mathcal{S}$ as well. Substituting (2.5) into the formula for $\tilde{f}$, we obtain

$$\begin{aligned}\tilde{f}(\sigma,\omega) &= \frac{(-i\sigma)^{(n-3)/2}}{(2\pi)^{n/2}} \int \Big(\nabla f_1 \cdot (i\sigma\omega) + f_2\Big) \exp(i\sigma\omega \cdot x) dx \\ &= (-i\sigma)^{(n-3)/2}\Big((-i\sigma)^2|\omega|^2\hat{f}_1(\sigma\omega) + \hat{f}_2(\sigma\omega)\Big).\end{aligned}$$

Here, $\hat{f}$ denotes Fourier transform. Since the Fourier transform maps $\mathcal{S}$ to itself, and $\mathcal{S}$ is closed under multiplication by polynomials, we see that $\tilde{f} \in \mathcal{S}$.

We set $h = A_o f$, and proceed by computing the representation of h. Recalling the definition of A_o and using the fact that $\phi_{\sigma,\omega}$ is an eigenfunction with eigenvalue $i\sigma$, we get after an integration by parts

$$\tilde{h}(\sigma,\omega) = (A_o f, \phi_{\sigma,\omega})_E = -(f, A_o\phi_{\sigma,\omega})_E = i\sigma(f, \phi_{\sigma,\omega})_E = i\sigma\tilde{f}.$$

We see that this representation takes A_o to multiplication by $i\sigma$, so we indeed have a spectral representation for A_o. Next, we show that it is a spectral representation also for $\{U_o(t)\}$.

Recall first that $U_o(t)f = \{u(x,t), u_t(x,t)\}$, where u is the solution to the wave equation with initial data f. For an eigenfunction $\phi_{\sigma,\omega}$, we get $U_o(t)\phi_{\sigma,\omega} = e^{-i\sigma(\omega\cdot x - t)}(1, i\sigma)^T = e^{i\sigma t}\phi_{\sigma,\omega}$. Using this and the fact that $U_o(t)$ is a unitary operator, we get for the solution u to initial data f,

$$\tilde{u}(\sigma,\omega,t) = (U_o(t)f, \phi_{\sigma,\omega})_E = (f, U_o(-t)\phi_{\sigma,\omega})_E = e^{i\sigma t}(f, \phi_{\sigma,\omega}) = e^{i\sigma t}\tilde{f},$$

so we have a representation for the group, such that $U_o(t)$ is mapped into multiplication by $e^{i\sigma t}$.

Next, we show that the representation (2.6) is isometric. We start out with an integration by parts in the energy scalar product which yields

$$\begin{aligned}(f,\phi_{\sigma,\omega})_E &= \frac{1}{2}\int \nabla f_1\cdot\nabla\bar{\phi}_{\sigma,\omega,1} + f_2\bar{\phi}_{\sigma,\omega,2}\,dx = \frac{1}{2}\int -f_1\Delta\bar{\phi}_{\sigma,\omega,1} + f_2\bar{\phi}_{\sigma,\omega,2}\,dx\\ &= \frac{1}{2}\int -f_1(x)(i\sigma)^2 e^{i\sigma\omega\cdot x} + f_2(x)(-i\sigma)e^{i\sigma\omega\cdot x}dx.\end{aligned}$$

Using this in the definiton (2.6) of $\tilde{f}$, we get

$$\begin{aligned}\tilde{f} &= \frac{1}{2}\frac{1}{(2\pi)^{(n/2)}}\int -(-i\sigma)^{(n+1)/2}f_1(x)e^{i\sigma\omega\cdot x} + (-i\sigma)^{(n-1)/2}f_2(x)e^{i\sigma\omega\cdot x}dx\\ &= -(-i\sigma)^{(n+1)/2}\tilde{f}_1 + (-i\sigma)^{(n-1)/2}\tilde{f}_2, \end{aligned}\tag{2.7}$$

with

$$\tilde{f}_j = \frac{1}{2}\frac{1}{(2\pi)^{(n/2)}}\int f_1(x)e^{i\sigma\omega\cdot x}dx = \frac{1}{2}\hat{f}_j(\sigma\omega), \qquad j=1,2. \tag{2.8}$$

We see from (2.8) that $\tilde{f}_j$, $j=1,2$ are even functions in (σ,ω). Therefore one of the two summands on the right-hand side of (2.7) is even, and the other is odd, whence they are orthogonal in $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$, and we find that

$$\|\tilde{f}\|^2_{L^2} = \|(-i\sigma)^{(n+1)/2}\tilde{f}_1\|^2_{L^2} + \|(-i\sigma)^{(n-1)/2}\tilde{f}_2\|^2_{L^2}. \tag{2.9}$$

Furthermore, since $\tilde{f}_j$ are even functions, $|\sigma^k\tilde{f}_j|$ are also even, and we get

$$\|(-i\sigma)^{(n+1)/2}\tilde{f}_1\|^2_{L^2} = 2\int_{\mathbb{S}^{n-1}}\int_0^\infty |\tilde{f}_1|^2\sigma^{n+1}d\sigma d\omega,$$

analogously for $\|(-i\sigma)^{(n-1)/2}\tilde{f}_2\|^2_{L^2}$. Now we use (2.8) and Plancherel's Theorem to obtain

$$\|(-i\sigma)^{(n+1)/2}\tilde{f}_1\|^2_{L^2} = 2\int_{\mathbb{S}^{n-1}}\int_0^\infty |\frac{1}{2}\hat{f}_1(\sigma\omega)|^2\sigma^{n+1}d\sigma d\omega = \frac{1}{2}\int|\hat{f}_1(\xi)|^2|\xi|^2d\xi = \frac{1}{2}\int|\nabla f_1(x)|^2dx,$$

and in the same way

$$\|(-i\sigma)^{(n-1)/2}\tilde{f}_2\|^2_{L^2} = \frac{1}{2}\int|f_2(x)|^2dx.$$

The last two identities together with (2.9) show that we have the isometry of the representation,

$$\|\tilde{f}\|^2_{L^2} = \|f\|^2_E.$$

To show that the representation is in fact unitary, it remains to show that the set of representations of data in $\mathcal{S}^2$ is dense in $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$. To see this, recall that the fact that the two functions on the right-hand side of (2.7) are of different parity. Consequently, if $\tilde{f}$ is smooth and vanishes for σ near zero and near infinity, both $\tilde{f}_1$ and $\tilde{f}_2$ are smooth with compact support, and using (2.8), this holds for $\hat{f}_1$ and $\hat{f}_2$, such that we can conclude that f_1 and f_2 belong to $\mathcal{S}$. So all functions $\tilde{f}$ that are smooth and compactly supported in σ represent $\mathcal{S}$ data. But these functions are dense in $L^2(\mathbb{R}\times\mathbb{S}^{n-1})$, so this completes the proof. □

The inverse Fourier transform with respect to σ of the spectral representation for f yields

$$k(s,\omega) = \mathcal{F}^{-1}_\sigma[\tilde{f}](s,\omega) = \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^\infty \tilde{f}(\sigma,\omega)e^{-i\sigma s}d\sigma \in L^2(\mathbb{R}\times\mathbb{S}^{n-1}),$$

which is the translation representation of f. The following theorem gives the explicit relation between a Schwartz function and its translation representation.

Theorem 2.5. Let $f \in \mathcal{S}^2$, and denote by k its translation representation, i.e., k is obtained from the spectral representation $\tilde{f}$ by inverse Fourier transformation. Then k is expressed in terms of f as

$$k(s,\omega) = -\partial_s^{(n+1)/2} M_1(s,\omega) + \partial_s^{(n-1)/2} M_2(s,\omega), \tag{2.10}$$

with M_j, $j = 1, 2$, defined as integrals over the hyperplanes $x \cdot \omega = s$

$$M_j(s,\omega) = \frac{1}{2}\frac{1}{(2\pi)^{(n-1)/2}} \int_{x\cdot\omega = s} f_j(x) dS.$$

Conversely, we can express f_j, $j = 1, 2$, as the following integrals over spheres:

$$f_1(x) = S(x) := \int_{\mathbb{S}^{n-1}} h(x\cdot\omega, \omega) d\omega, \qquad f_2(x) = S'(x) := \int_{\mathbb{S}^{n-1}} h'(x\cdot\omega,\omega) d\omega, \tag{2.11}$$

with

$$h(s,\omega) = \frac{1}{(2\pi)^{(n-1)/2}} (-\partial_s)^{(n-3)/2} k(s,\omega), \qquad h'(s,\omega) = \frac{1}{(2\pi)^{(n-1)/2}} (-\partial_s)^{(n-1)/2} k(s,\omega). \tag{2.12}$$

Proof. Let $f \in \mathcal{S}^2$. Recalling the expressions (2.8), we split the integration and first integrate along the hyperplane $x \cdot \omega = s$, and then along $s \in \mathbb{R}$, to get for $j = 1, 2$

$$\tilde{f}_j(\sigma,\omega) = \frac{1}{2}\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} \int_{x\cdot\omega=s} f_j(x) dS e^{i\sigma s} ds = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} M_j(s,\omega) e^{i\sigma s} ds = \hat{M}_j(\sigma,\omega),$$

by definition of M_j. Substituting this into (2.7), we get

$$\begin{aligned} \tilde{f}(\sigma,\omega) &= -(-i\sigma)^{(n+1)/2} \hat{M}_1(\sigma,\omega) + (-i\sigma)^{(n-1)/2} \hat{M}_2(\sigma,\omega) \\ &= \mathcal{F}_s\big[- \partial_s^{(n+1)/2} M_1(s,\omega) + \partial_s^{(n-1)/2} M_2(s,\omega)\big], \end{aligned}$$

where $\mathcal{F}_s$ denotes Fourier transform in the variable s. So the inverse Fourier transform of $\tilde{f}$ in the variable σ is indeed of the form (2.10).

To prove the second part of the theorem, note that the translation representation is unitary (recall that the scalar product on $L^2(\mathbb{R} \times \mathbb{S}^{n-1})$ is denoted by square brackets),

$$(f, g)_E = [k, l], \tag{2.13}$$

if $f, g \in H_o$ and k, l are the respective translation representations. This is true since the spectral representation, as well as the Fourier transform on $L^2(\mathbb{R} \times \mathbb{S}^{n-1})$ are both unitary. Using density, it is sufficient to assume $g \in \mathcal{S}^2$. Thus, we can use the first part of the theorem and express l as in (2.10), and then integrate by parts to obtain

$$\begin{aligned} [k, l] &= \int\int k(s,\omega)\big(- \partial_s^{(n+1)/2} \bar{M}_1^l(s,\omega) + \partial_s^{(n-1)/2} \bar{M}_2^l(s,\omega)\big) ds\, d\omega \\ &= (2\pi)^{(n-1)/2} \int\int h_1(s,\omega) \bar{M}_1^l(s,\omega) + h_2(s,\omega)\bar{M}_2^l(s,\omega) ds\, d\omega, \end{aligned}$$

with

$$h_1(s,\omega) = -\frac{1}{(2\pi)^{(n-1)/2}} (-\partial_s)^{(n+1)/2} k(s,\omega), \qquad h_2(s,\omega) = \frac{1}{(2\pi)^{(n-1)/2}} (-\partial_s)^{(n-1)/2} k(s,\omega).$$

Now we use the definiton of M_j and recombine integration over the hyperplane and integration with respect to s to integration with respect to x, which yields

$$\begin{aligned}[k,l] &= \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}} h_1(s,\omega)\left(\int_{x\cdot\omega=s}\bar{g}_1(x)dS\right) + h_2(s,\omega)\left(\int_{x\cdot\omega=s}\bar{g}_2(x)dS\right) ds\,d\omega \\ &= \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}^n} h_1(x\cdot\omega,\omega)\bar{g}_1(x) + h_2(x\cdot\omega,\omega)\bar{g}_2(x)dx\,d\omega,\end{aligned}$$

which after exchanging the order of integration becomes

$$[k,l] = \frac{1}{2}\int_{\mathbb{R}^n} S_1(x)\bar{g}_1(x) + S_2(x)\bar{g}_2(x)\,dx, \tag{2.14}$$

with

$$S_j(x) = \int_{\mathbb{S}^{n-1}} h_j(x\cdot\omega,\omega)\,d\omega.$$

For the left-hand side of (2.13), we integrate by parts in the first term to get

$$(f,g)_E = \frac{1}{2}\int_{\mathbb{R}^n} -\Delta f_1\bar{g}_1 + f_2\bar{g}_2\,dx. \tag{2.15}$$

In view of (2.13), the right-hand sides of (2.14) and (2.15) must be equal for all $g \in \mathcal{S}^2$. Since Schwartz class data are dense in H_o, we infer

$$-\Delta f_1 = S_1, \quad \text{and} \quad f_2 = S_2. \tag{2.16}$$

So the second identity of (2.11) is proved. To obtain the first identity, note that

$$S_1 = -\Delta S$$

for the function S defined in (2.11)-(2.12), which, combined with the first of the equations (2.16), gives $\Delta(f_1 - S) = 0$, so $f_1 - S$ is a globally harmonic function. But $f \in \mathcal{S}^2$, and thus the definition of S shows that $S \to 0$ as $|x| \to \infty$, hence $f_1 - S \to 0$ as $|x| \to \infty$, and by the maximum principle for harmonic functions, we find that $f_1 - S$ vanishes identically, and we have obtained the first of (2.11). □

Corollary 2.6. Let $u(x,t)$ be the solution to the wave equation (2.1)-(2.2) with initial data $f \in H_o$, and assume that the translation representation k of f is smooth. Then

$$u(x,t) = \int_{\mathbb{S}^{n-1}} h(x\cdot\omega - t,\omega)\,d\omega, \quad \text{and} \quad u_t(x,t) = \int_{\mathbb{S}^{n-1}} h'(x\cdot\omega - t,\omega)\,d\omega. \tag{2.17}$$

Proof. This follows by noting that $U_o(t)$ is now represented by right translation: let $f \in H_o$ and u be the corresponding solution to the wave equation. Using that the spectral representation of $U_o(t)$ is multiplication by $e^{i\sigma t}$, we compute

$$\mathcal{F}_\sigma^{-1}[\tilde{u}](s,\omega,t) = \frac{1}{(2\pi)^{n/2}}\int e^{i\sigma t}\tilde{f}(\sigma,\omega)e^{-i\sigma s}d\sigma = \mathcal{F}_\sigma^{-1}[\tilde{f}](s-t,\omega) = k(s-t,\omega).$$

Now apply the expressions (2.11) to $U_o(t)f$. □

This translation representation relates to the outgoing and incoming subspaces $D_\pm$ introduced in Definition 2.2 in the following way. For a proof, see [1, Sec.IV, Thm. 2.3].

Theorem 2.7. The subspaces $L^2((-\infty,0)\times\mathbb{S}^{n-1})$ and $L^2((0,\infty)\times\mathbb{S}^{n-1})$ associated with the translation representation established above are the incoming and outgoing subspaces D_- and D_+, respectively. From this we see in particular that D_- and D_+ are orthogonal. □

3 The wave equation in an exterior domain

We now turn to the case of the presence of an obstacle: We consider the wave equation in an exterior domain $G \subset \mathbb{R}^n$, that is, the complement of a closed bounded set, at the boundary of which the solution to the wave equation must vanish. For some properties and estimates for solutions to this problem, we refer the reader to [1, Sec. V].

Before we proceed to construct the group $\{U(t)\}$ for this problem, to which we can apply the tools developed in Section 1, we introduce some notation.

As initial data, we will have pairs of complex-valued functions $f = \{f_1, f_2\}$ defined in G, for which the energy norm

$$\|f\|_E^2 = \frac{1}{2}\int_G |\nabla f_1|^2 + |f_2|^2\,dx \tag{3.1}$$

is finite. We denote by H the completion of $C_c^\infty(G)^2$ with respect to this norm. Note that this space embeds as a subspace into H_o, if we define data in H to be zero outside G.

We also define the space H_D as the closure of $C_c^\infty(G)$ in the Dirichlet norm, given by

$$\|u\|_D^2 = \int_G |\nabla u|^2 dx.$$

With this, the energy norm becomes $\|f\|_E^2 = \|f_1\|_D^2 + \|f_2\|_{L^2(G)}^2$. Finally, for a subdomain $G' \subset G$, we denote by $\|f\|_{E,G'}$ the local energy obtained by integrating only over G' in (3.1).

3.1 The group $\{U(t)\}$

We will now construct the operators $U(t)$, which as before will assign to initial data f the corresponding solution of the wave equation in G at time t. We will do so in the opposite direction as compared to the previous section, by first constructing the operator A which will serve as infinitesimal generator for the group, and then applying Stone's Theorem.

Define the operator A as

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix},$$

with the domain $D(A)$ consisting of all pairs of data $f = \{f_1, f_2\}$ for which $Af = \{f_2, \Delta f_1\} \in H$. That is, $f_2 \in L^2(G) \cap H_D$, and Δf_1 (defined in the distributional sense) belongs to $L^2(G)$.

We want A to be generator of a unitary group; to facilitate this, we need the following

Theorem 3.1. The operator A is skew-selfadjoint.

Proof. We need to show that $A^* = -A$, and $D(A)$ is dense in H. The latter follows immediately from the fact that $D(A)$ contains all data in $C_c^\infty(G)^2$. For the former property, we start by showing that A is skew-symmetric, hence, A^* is an extension of the operator $-A$. Let $f \in D(A)$, and consider first $g \in C_c^\infty(G)^2$. Then an integration by parts yields

$$(Af, g)_E = \frac{1}{2}\int_G \nabla f_2 \cdot \nabla \bar{g}_1 + \Delta f_1 \bar{g}_2\,dx = \frac{1}{2}\int_G \nabla f_2 \cdot \nabla \bar{g}_1 - \nabla f_1 \cdot \nabla \bar{g}_2\,dx = -(f, Ag)_E. \tag{3.2}$$

For arbitrary $g \in D(A)$, we have $g_2 \in L^2(G) \cap H_D$, and thus we can find a sequence $\{g_m\} \in C_c^\infty(G)^2$ that converges to g in the H norm (since this space is dense in H)and for which also $g_{m,w} \to g_2$ in H_D. So (3.2) holds for all $g \in D(A)$, and we find that A is skew-symmetric. In order to show that in fact $A^* = -A$, we recall the definition of the adjoint. Let $g \in D(A^*)$, and denote $h = A^*g$. We need to show that $g \in D(A)$ and $h = -Ag$. By definition of A^*, for $f \in D(A)$,

$$(Af, g)_E = (f, h)_E. \tag{3.3}$$

By taking f with $f_1 = 0$ and $f_2 \in C_c^\infty(G)$, this identity becomes

$$\frac{1}{2}\int_G \nabla f_2 \cdot \nabla \bar{g}_1 \, dx = \frac{1}{2}\int_G f_2 \bar{h}_2 \, dx,$$

which after integration by parts on the left-hand side yields

$$(\Delta f_2, g_1)_{L^2(G)} = (f_2, h_2)_{L^2(G)}, \tag{3.4}$$

whence we conclude that $-\Delta g_1 = h_2$ in the sense of distributions.

If we now take f with $f_2 = 0$, then (3.3) yields

$$(\Delta f_1, g_2)_{L^2(G)} = (f_1, h_1)_D. \tag{3.5}$$

We choose $f_1 \in H_D$ to plug into this identity by means of the Riesz representation theorem as follows: Fix an arbitrary $\phi \in C_c^\infty G$, and let G' be the support of G; in particular, G' is compact. By (**??**), if $\psi \in H_D$, then $\|\psi\|_{L^2(G')}^2 \leq C\|\psi\|_D^2$. Using the Cauchy-Schwarz inequality, this implies

$$|(\phi, \psi)_{L^2(G)}| \leq C\|\psi\|_D,$$

so the liner functional $l(\psi) = \overline{(\phi, \psi)_{L^2(G)}}$ is bounded in the Dirichlet norm. So we can employ the Riesz representation theorem, which guarantees the existence of $f_1 \in H_D$ such that

$$(f_1, \psi)_D = (\phi, \psi)_0 \tag{3.6}$$

for $\psi \in H_D$. If $\psi \in C_c^\infty(G)$, we can integrate by parts to obtain

$$-(\Delta f_1, \psi)_{L^2(G)} = (\phi, \psi)_{L^2(G)},$$

so we have $-\Delta f_1 = \phi$ in the sense of distributions. Using this on the left-hand side of (3.5), we get

$$-(\phi, g_2)_{L^2(G)} = (f_1, h_1)_D = (\phi, h_1)_{L^2(G)},$$

where the last equation was obtained using (3.6). Thus, $-g_2 = h_1$. This, together with (3.4), shows that $g = \{g_1, g_2\}$ lies in $D(A)$ and $h = A^* g = -Ag$, which completes the proof. □

Now we can use Stone's Theorem, which states that A generates a one-parameter group $\{U(t)\}$ of unitary operators, with the following properties:

(i) $U(t)$ is strongly continuous in t;

(ii) $U(t)f$ is strongly differentiable with respect to t if and only if $f \in D(A)$, and in this case

$$\frac{d}{dt}U(t)f = AU(t)f; \tag{3.7}$$

(iii) $U(t)$ maps $D(A)$ onto itself and commutes with A.

Note that if $f \in D(A)$ and $u(x,t) = U(t)f$, then the second component of (3.7) shows that

$$u_{tt} = \Delta u$$

in the distributional sense, so u is a distribution solution to the wave equation with initial data f.

3.2 Outgoing and incoming subspaces and the scattering operator

We will now define the outgoing and incoming subspaces for $\{U(t)\}$. To this end, we fix $\rho > 0$ such that $\{|x| < \rho\}$ contains ∂G, and set $D_+^\rho = U_o(\rho)D_+$ and $D_-^\rho = U_o(\rho)D_-$, where $D_\pm$ are the outgoing and incoming subspaces for the free space problem defined in Section 2.

Note that it follows from the definition of D_+ that if $f \in D_+^\rho$, $U(t)f$ vanishes in the truncated forward cone $\{|x| < t + \rho\}$ for $t > 0$; analogously, if $f \in D_-^\rho$, then $U(t)f$ vanishes in the truncated backward cone $\{|x| < -t + \rho\}$ for $t < 0$.

These spaces are in fact outgoing and incoming according to Definition 1.1. For a proof, see [1, Sec. V, Thm. 2.1].

Theorem 3.2. D_+^ρ is an outgoing subspace, i.e., D_+^ρ is a closed subspace with the following properties:

(i) $U(t)D_+^\rho \subset D_+^\rho$ for $t > 0$;

(ii) $\bigcap U(t)D_+^\rho = \{0\}$;

(iii) $\overline{\bigcup U(t)D_+^\rho} = H$.

Also, D_-^ρ is an incoming subspace, and D_+^ρ and D_-^ρ are orthogonal. □

Now the results from Section 1 apply to our situation and secure the existence of an outgoing translation representation for $\{U(t)\}$ by $L^2(-\infty, \infty; N)$ with some auxiliary Hilbert space N (unique up to isomorphism), under which D_+^ρ gets mapped onto $L^2(0, \infty; N)$.

We want to relate the outgoing translation representation for $\{U(t)\}$ to the translation representation for $\{U_o\}$ constructed in the previous section. We first note that by our choice of ρ, for $t > 0$, $U_o(t)$ and $U(t)$ act in the same way on the space D_+^ρ. Furthermore, by definition of $D_+^\rho = U_o(\rho)D_+$, we find that in the translation representation for $\{U_o(t)\}$, D_+^ρ maps onto $L^2(\rho, \infty; N)$.

Now suppose that $f \in D_+^\rho$ gets mapped to k_o in the free space translation representation (which we now consider as a function of one real variable t with values in the Hilbert space N). Then $U_o(t)f$ gets mapped to $k_o(s-t)$. We will find the corresponding outgoing translation representation for f. First note that the map

$$f \mapsto k_+(s) = k_o(s + \rho)$$

maps D_+^ρ onto $L^2(0, \infty; N)$, such that $U(t)f = U_o(t)f$ is taken to $k_o(s-t+\rho) = k_+(s-t)$ for $t > 0$, so

$$U(t)f \mapsto k_+(s - t), \tag{3.8}$$

i.e., $U(t)$ acts as right shift on D_+^ρ. Now recall that data f is called eventually outgoing if there is some r such that $U_o(r)f \in D_+$. Then $U_o(r + \rho)f \in D_+^\rho$. Extending the above mapping to all such f while preserving (3.8), this yields an outgoing translation representation for $\bigcup U(t)D_+^\rho$, which we can further extend by continuity to the closure of this set, which is H by Theorem 3.2. So we have an outgoing translation representation for $\{U(t)\}$ on H, given by (3.8).

Now we define the wave operators $W_\pm$, which in turn will be used to define the scattering operator for scattering by an obstacle.

Let

$$W_\pm = \text{strong} \lim_{t \to \pm\infty} U(-t)U_o(t). \tag{3.9}$$

Note that if $f \in D_+^\rho$, then by the above observation, $U(t)f = U_o(t)f$, and hence

$$U(-t)U_o(t)f = f \tag{3.10}$$

for $t > 0$, so W_+ acts as the identity on D_+^ρ. Also, if f is eventually outgoing, then for some $T > 0$, $U(T)f \in D_+^\rho$, and using (3.10), we find that for $t > 0$,

$$U(-T-t)U_o(T+t)f = U(-T)U(-t)U_o(t)U_o(T)f = U(-T)U_o(T)f,$$

whence $W_+ f = U(-T)U_o(T)f$. Considering W_+ as a map from the free space translation representation k_o of f to the outgoing translation representation, this corresponds to

$$k_o(s) \mapsto k_o(s+\rho) = k_+(s), \tag{3.11}$$

for all eventually outgoing f. Note that for the translation representation, the property of being eventually outgoing corresponds to k_o vanishing for sufficiently small s, say, $s < T$, since eventually outgoing means that for some T, $U_o(T)f \in D_+$, and $U_o(t)$ goes into right translation. Now property (iii) of the definition of an outgoing subspace implies that eventually outgoing data are in fact dense in H_o: $U_o(r)f \in D_+$ implies $f \in U(-r)D_+$, and conversely, data $f \in U(r)D_+$ are eventually outgoing as $U(-r)f \in D_+$. Therefore, W_+ has the representation (3.11) for all $f \in H_o$, and since the mapping (3.11) is onto, we can also conclude that the range of W_+ is all of H.

Analogously one can show that W_- corresponds to the mapping

$$k_o(s) \mapsto k_-(s) = k_o(s-\rho),$$

taking the free space translation representation to the incoming translation representation for $\{U(t)\}$, and W_- maps H_o onto H.

This allows us to define the *scattering operator* S as

$$S = W_+^{-1}W_-, \tag{3.12}$$

which is then a well-defined unitary map from H_o to itself. In terms of the incoming and outgoing representations, S can be realized as the map

$$k_-(s+\rho) \mapsto k_+(s-\rho),$$

with $k_-(s)$ the incoming, $k_+(s)$ the outgoing translation representers of f.

Thus, S relates the "initial state"of the solution to the wave equation, i.e., starting out near $t = -\infty$, to the "final state", near $t = \infty$, after the scattering process. In many physical applications, these are data easily accessible by measurement, and the importance of the scattering operator that encodes these data is in providing a means to study the otherwise inaccessible obstacle: the goal is to gain information about the scattering object by studying S.

An important question in this context is whether the scattering operator uniquely determines the obstacle, such that from measurements of the scattered data one may reconstruct the scatterer. This is the *inverse scattering problem*, and the machinery developed here that led to the definition of the scattering operator S is useful in deriving many properties of S that help answer this question. It turns out that in our case, the answer is affirmative:

Theorem 3.3. The scattering operator uniquely determines the scatterer.

For a proof (as well as for a thorough study of S using the construction presented here), we refer the reader to Theorem 5.6 in Section V of [1], where two different flavored proofs are presented, one using an integral operator representation of S, the other using the wave operators $W_\pm$ and the group $\{U(t)\}$.

References

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