ANALYTIC TORSION

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Abstract. This article is a survey of analytic torsion of elliptic operator complexes. The scope of the article is expository. We define the analytic torsion of elliptic complexes in general and with this define the analytic torsion in terms of the spectrum of the Laplace operator on a Riemannian Manifold. We define the Riedemeister Torsion of a Manifold and discuss the celebrated Cheeger-Mueller theorem relating analytic torsion and Riedemeister torsion.

0. Introduction

Analytic Torsion is an invariant of Riemannian Manifolds, first introduced by Ray and Singer in 1970s. It is defined in terms of determinants of Laplacians on n-forms of a Riemannian Manifold M. Using same construction it is defined for general elliptic complexes. Analytic torsion was introduced as the analytic version of the Reidemeister torsion (R-torsion).

R-torsion is an algebraic topology invariant introduced by Reidemeister in 1935 [1] in classification of three dimensional lens spaces using simplicial chain complex of universal covers. Lens spaces were first known examples of 3-manifolds whose homotopy type and homology do not determine their homeomorphism type. R-torsion was generalized by Franz [2] in same year, to higher dimensional lens spaces. There is a complete classification of 3dimensional lens spaces in terms of R-torsion and fundamental group. There exist homotopic lens space with different R-torions. Though R-torsion is not a homotpy invariant, it was shown to be a topological invariant by Kirby and Siebenmann in 1969.

In 1971 D.B. Ray and I.M. Singer conjectured the equality of the two torsions, which was proved independently J. Cheeger and W. Muller later same year.

Here is an overview of the sections in the article.

In the first section we discuss some preliminary topics needed to define the analytic and R-torsions. The notion of a torsion associated with a general finite Hilbert-chain complex are defined and some parallels with similar definition for torsion of an elliptic complex are discussed.

Second section established the notion of analytic torsion of a Riemannaian Manifold. This definition relies on zeta regularized determinants of the Laplacian of an elliptic complex.

Third section builds on the definition of torsion of a Hilbert-chain complex from section two, to Hilbert-chain complex associated with the $\mathbb{Z}[\pi_0(M)]$-action on the cellular chain complex of finite CW-complexes.

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Contents

1. Some Preliminaries

Definition 1.1. Hilbert Chain Complex (HCC) A HCC is a chain complex $C_* = (C_*, d_*)$ consisting of Hilbert Spaces C_n and linear maps $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots$$

As usual we can define the homology $H(C_*) = ker(d_n)/im(d_{n+1})$. Each homology is also a Hilbert space with the corresponding quotient metric. The Lapalace operator can be definded as:

$$\Delta_n = c_n^* \circ c_n + c_{n+1} \circ c_{n+1}^* : C_n \to C_n$$

Definition 1.2. Elliptic Complex Let us now consider hermitian vector bundles E_k $k = 0, 1, ..., n$ over a manifold M and, for each k, let $D_k : \Gamma(E_k) \to \Gamma(E_{k+1})$ be classical differential operators of the same positive order N. Then the complex

$$0 \xrightarrow{D_0} \Gamma(E_0) \xrightarrow{D_1} \cdots \to \Gamma(E_k) \xrightarrow{D_k} \Gamma(E_{k+1}) \xrightarrow{D_{k+1}} \cdots \xrightarrow{D_n} \Gamma(E_n) \to 0$$

is an elliptic complex if each D_k is a differential operator and $D_{k+1} \circ D_k = 0$ such that the associated symbol sequence is exact.

$$0 \xrightarrow{D_0} \pi_0^*(E_0) \xrightarrow{D_1} \cdots \to \pi_0^*(E_k) \xrightarrow{D_k} \pi_0^*(E_{k+1}) \xrightarrow{D_{k+1}} \cdots \xrightarrow{D_n} \pi_0^*(E_n) \to 0$$

where $\pi_0^* : T_0^*(M) \to M$ is the projection from tangent bundle without the zero section.
The Laplacian is again defined as before.

2. Analytic Torsion

Definition 2.1. Zeta regulaized determinnats Let (E_*, d_*) be an Elliptic complex on a closed manifold M. The n-th Zeta function is defined for $z \in \mathbb{C}$ by

$$\zeta_n(z) = \sum_{\lambda_j \geq 0} \lambda_j^{-z}$$

where $\{\lambda_j | j \in I\}$ are eigen values of Δ_n counted with multiplicities. It is not clear if this sum converges but following regularization of ζ_n is given

Lemma 2.2. *Regularity of zeta function The Zeta-function ζ_n converges absolutely for $s \in \mathbb{C}$ with $\Re(s) \geq \dim(M)/2$ and defines a holomorphic function. It has a meromorphic extension to $\mathbb{C}$ which is analytic at 0 and $\frac{d}{ds}\Big|_{s=0} \zeta_n(s) \in \mathbb{R}$.*

Definition 2.3. (Analytic Torsion) Let M be a closed Riemannnian Manifold. Its analytic torsion is defined as

$$\ln(\tau_{an}) = \frac{1}{2} \sum_{n \geq 0} (-1)^n \cdot n \cdot \frac{d}{ds}\Big|_{s=0} \zeta_n(s)$$

Proof. (Sketch of Lemma 2.2) Sketch of proof follows closely as in (M. P. Gilkey,[3])

For a elliptic self adjoint positive ΨDO P we can formally define

$$P^{-z} = \frac{1}{2\pi i} \int_C \lambda^{-z}(P - \lambda) d\lambda$$

where C is a suitable curve in half plane $\Re(\lambda) > 0$. For $\Re(z) >> 0$,

$$K(z, x, y) = \sum_j \lambda_j^{-z} \ \phi_j(x) \otimes \phi_j^*(y)$$

and therefore

$$Tr(P^{-z}) = \sum_j \lambda_j^{-z}$$

Allowing us to rewrite the n-th zeta function as:

$$\zeta_n(z) = Tr(P^{-z})$$

Using Mellin Transform we can relate the trace of P^{-z} to the trace of associated heat operator

$$\int_0^\infty t^{z-1} e^{-\lambda t} dt = \lambda^{-z} \int_0^\infty (\lambda t)^{z-1} e^{-\lambda t} d(\lambda t) = \lambda^{-z} \Gamma(z)$$

so we can write

$$\Gamma(z) Tr(P^{-z}) = \int_0^\infty t^{z-1} Tr(e^{-tP}) dt$$

Using the following asymptotic expansion of the trace of heat kernel at $t = 0$, (see lemma 7.18 Melrose [4])

$$\sum_{j=0}^{\infty} a_j \ t^{-\frac{n}{2}+j} \quad, a_j = 0 \ for \ j \ even$$

where $n = \dim(M)$. and so the integral

$$\int_0^\infty t^{-z-1} Tr(e^{-tP}) dt$$

converges absolutely for $\Re(z) > \frac{n}{2}$ and extends to a meromorphic function with poles at $z = j - \frac{n}{2}$.

Finally we can rewrite the n-th zeta function as

$$\zeta_n(z) = \frac{z}{\Gamma(z)} \int_0^\infty t^{-z-1} Tr(e^{-tP}) dt$$

Definition 2.4. (Zeta Regularized Determinant) Using the relation between finite dimensional trace and determinant, we can define the formal determinant of P in terms of the ζ_n as follows

$$\det(P) = \exp(-\zeta_n'(P,0))$$

2.1. **Analytic Torsion of S^1.** We will compute the analytic torsion of S^1. Let the metric on S^1 be the angle t. The Laplacian is given by

$$\Delta_1 = d \circ d^* \quad , \Delta_1(f(t)dt) = -f''(t)dt$$

$$d : f(t) \mapsto f'(t)dt \quad d^* : g(t)dt \mapsto -g'(t)$$

The 2π periodic solutions of

$$f''(t) = -\lambda f(t)$$

give the eigen spaces of $\Delta_1 : \Omega^1(S^1) \to \Omega^1(S^1)$

$$E_{\lambda_j} = \begin{cases} \text{span}\{\cos(jt)dt,\ \sin(jt)dt\} & if \lambda_j = j^2 \\ \text{span}\{\text{dt}\} & \text{if } \lambda_j = 0 \end{cases}$$

Now we can write the 1^{st}-zeta function as

$$\zeta_1(z) = \sum_{n=1}^{\infty} (\lambda^2)^{-z}$$

Using results for the Riemannian Zeta Function

$$\zeta(z) = \sum_{n=1}^{\infty} \lambda^{-z}$$

and

$$\zeta(0) = -\frac{1}{2} \quad , \quad \zeta'(0) = -\frac{\ln(2\pi)}{2}$$

Comparing ζ_1 and ζ we see $\zeta_1(z) = \zeta(2z)$.

Finally we can write the analytic torsion as

$$\begin{aligned}
\tau_{an}(S^1) &= \frac{1}{2}\sum_n (-1)^n \cdot n \cdot \frac{d}{ds}\bigg|_{s=0} \zeta_n(s) \\
&= -\frac{1}{2} \cdot \frac{d}{ds}\bigg|_{s=0} \zeta_1(s) \\
&= -\frac{1}{2} \cdot \frac{d}{ds}\bigg|_{s=0} \zeta(2s) \\
&= -\frac{1}{2} \cdot -\frac{\ln(2\pi)}{2} \\
&= \ln(2\pi)
\end{aligned}$$

3. Riedemeister Torsion

Definition 3.1. (Torsion of acyclic chain complex) A chain complex over a module A is acyclic if

$$H_r(C) = 0 \ r \geq 0$$

It is a standard result from Homological Algebra that a chain complex (C_*, d_*) is cyclic iff there exists a chain contraction

$$\Gamma : 0 \cong 1 : C \to C$$

where each map $\Gamma_r : C_r \to C_{r+1}$ satisfies

$$d\Gamma + \Gamma d \ = \ 1 \ : C_r \to C_r$$

Lemma 3.2. *If C is an acyclic A-module chain complex with a chain contraction Γ then the map $f = d + \Gamma \ : C_{odd} \to C_{even}$ is an isomorphism- with*

$$C_{odd} = C_1 \oplus C_3 \oplus C_5 \oplus \cdots \ , \ C_{even} = C_0 \oplus C_2 \oplus C_4 \oplus \cdots$$

Proof. (Sketch) Consider the map $g = \Gamma + d : C_{even} \to C_{odd}$. Then both maps

$$f \circ g : C_{even} \to C_{even} \quad , \quad g \circ f : C_{odd} \to C_{odd}$$

are invertible and therefore we can write $f^{-1} = (g \circ f)^{-1} \circ g$. Hence f is isomorphism

Definition 3.3. The Reidemeister torsion of an acyclic chain complex C of finite based free A-module is

$$\tau_{top} = \det\left(d + \Gamma : C_{odd} \to C_{even}\right)$$

3.1. **Acyclic chain complex of a CW complex X.** Generally the cellular-chain complex of a CW complex is not acyclic over the module $\mathbb{Z}$, however we can consider the cellular chain complex of a universal $\widetilde{X}$ cover of X as a $\mathbb{Z}[\pi_0(X)]$ module, where the group ring consists of finite formal sums

$$\sum_{g \in \pi_1(X)} n_g \cdot g \quad (n_g \in \mathbb{Z})$$

$\pi_1(X)$ acts on $\widetilde{X}$ by group of covering translations(Deck transformations)

$$\pi_1(x) \times \widetilde{X} \to \widetilde{X} : (g, x) \mapsto gx$$

Now the cellular complex of $\widetilde{X}$ can be seen

$$\cdots \to 0 \to C_n(\widetilde{X}) \xrightarrow{d_n} C_{n-1}(\widetilde{X}) \xrightarrow{d_{n-1}} C_{n-2}(\widetilde{X}) \to \cdots \xrightarrow{d_0} C_0(\widetilde{X}) \to 0$$

This chain complex is not acyclic as $H_0(\widetilde{X}) = \mathbb{Z}$. The key to define R-Torsion for a CW complex is to find a ring morphism $\phi : \mathbb{Z}[\pi_0(X)] \to A$ such that the induced A-module chain complex is acyclic

$$C(X; A) = A \otimes_{\mathbb{Z}[\pi_0(X)]} C(\widetilde{X})$$

Definition 3.4. (R-Torsion of a CW-complex) The Reidemeister torsion of a finite CW complex X with respect to a ring morphism $\phi : \mathbb{Z}[\pi_0(X)] \to A$ (A commutative ring) such that the induced complex $C_*(X; A)$ is acyclic, is

$$\tau_{top} = \ln(\det(d + \Gamma : C(X; A)_{odd} \to C(X; A)_{even}))$$

4. Cheeger-Mueller Theorem

Theorem 4.1. *(Cheeger-Muller Theorem) Let M be a closed Riemannian Manifold then*

$$\tau_{an}(M) = \tau_{top}(M) \tag{4.1}$$

This result first appeared as the Ray-Singer conjecture in 1971. Latter that year it was independently proved ny Muller and Cheeger. This was first proved for even dimensional manifolds and latter for odd dimensions.

By considering correct boundary conditions of the Laplace operator, this result can be extended to manifolds with boundary as done in [7] Corollary 5.1.

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