Partial Differential Equations Spring 2018: Exam 1 Solutions

Problem 1. Determine whether the following are TRUE or FALSE. You do *not* have to justify your answers. Correct answers are worth 2 points, incorrect answers are worth -2 points, and unanswered questions are worth 0. (Minimum score of 0).

(a) The equation
$$u_t + u_x = \sin u$$
is first order, linear, and inhomogeneous.

(b) If $u(x,t)$ is a solution to the wave equation $u_{tt} - c^2 u_{xx} = 0$ on $\mathbb{R} \times [0,\infty)$ with initial displacement $\phi(x)$ and velocity $\psi(x)$ both vanishing outside of the interval $[-a,a]$, then $u(a+2ct,t) = 0$.

(c) A solution $u(x,t)$ to the heat equation $u_t - ku_{xx} = 0$ on $[0,\ell] \times [0,\infty)$ may have a singularity at a point (x_0,t_0), where $0 < x_0 < \ell$, $0 < t_0$.

(d) The operator $-\frac{d^2}{dx^2}$ is self-adjoint with respect to the inner product $(f,g) = \int_0^1 f(x)\overline{g}(x)\,dx$, when acting on functions satisfying any boundary conditions of the form
$$\alpha_1 f(0) + \beta_1 f(1) + \gamma_1 f'(0) + \delta_1 f'(1) = 0,$$
$$\alpha_2 f(0) + \beta_2 f(0) + \gamma_2 f'(0) + \delta_2 f'(1) = 0.$$

(e) Energy is conserved for the wave equation.

(f) The heat equation is well posed on $\mathbb{R} \times \mathbb{R}$.

Solution.

(a) FALSE. The term $\sin(u)$ makes the equation nonlinear.

(b) TRUE. Finite propagation speed implies that u must vanish for $|x| > a + ct$.

(c) FALSE. Even with singular initial data, the solution to the heat equation is smooth (infinitely differentiable) for all $t > 0$.

(d) FALSE. Only for *symmetric boundary conditions* (wherein $(f'g + fg')\big|_{x=0}^{1} = 0$) is $-\frac{d^2}{dx^2}$ self-adjoint.

(e) TRUE. This is a fundamental feature of the wave equation.

(f) FALSE. The heat equation is only well-posed forward in time, not backward.

$\square$

Problem 2. Use the method of characteristics to solve the first order PDE
$$u_x + 2xu_y = 0, \qquad u(0,y) = \sin(y).$$

Solution. The characteristics of the equation satisfy the ODE
$$\frac{dy}{dx} = 2x,$$
and have solutions of the form
$$y(x) = x^2 + c$$
The general solution to $u_x + 2xu_y = 0$ is $u(x,y) = f(c) = f(y - x^2)$, and the additional condition $u(0,y) = \sin(y)$ implies $f = \sin$; thus $u(x,y) = \sin(y - x^2)$. $\square$

Problem 3. Use Duhamel's principle for the wave equation on $\mathbb{R}$ to solve the inhomogeneous equation with a source term:

$$u_{tt} - c^2 u_{xx} = 1,$$

subject to the initial conditions

$$u(x,0) = \sin(x), \qquad u_t(x,0) = 0.$$

Solution. The general solution to the equation $u_{tt} - c^2 u_{xx} = f$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$ is

$$u(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy + \frac{1}{2c}\iint_\Delta f(y,s)\, dy\, ds,$$

where Δ is the triangle of influence of (x,t), with vertices (x,t), $(x-ct,0)$, and $(x+ct,0)$. Thus in this case we have

$$u(x,t) = \frac{1}{2}\left(\sin(x-ct) + \sin(x+ct)\right) + \frac{1}{2c}\iiint_\Delta 1\, dy\, ds = \cos(ct)\sin(x) + t^2.$$

$\square$

Problem 4. Use the method of reflection to solve the wave equation on the half line $[0,\infty)$ with Dirichlet boundary condition:

$$u_{tt} - c^2 u_{xx} = 0, \qquad u(0,t) = 0,$$

and initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 1.$$

Solution. The solution is furnished by Del'Ambert's formula on $\mathbb{R}$ applied to the odd extension of the initial data:

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)\, dy,$$

where

$$\psi_{\text{odd}}(x) = \begin{cases} 1 & x > 0, \\ -1 & x < 0. \end{cases}$$

For $x > ct$, we obtain

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 1\, dy = t, \qquad x > ct,$$

while for $x < ct$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{0} -1\, dy + \frac{1}{2c}\int_0^{x+ct} 1\, dy \\ &= \frac{1}{2c}\left((x+ct) - (ct-x)\right) = \frac{x}{c}, \qquad x < ct. \end{aligned}$$

$\square$

Problem 5.

(a) On the interval $(0, \pi)$, determine the Fourier cosine series

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)$$

where

$$\phi(x) = \begin{cases} 1 & 0 < x < \frac{\pi}{2}, \\ 0 & \frac{\pi}{2} < x < \pi. \end{cases} \tag{1}$$

(b) Use your answer from part (a) to compute the limit of the series $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$

(c) Solve the heat equation $u_t - ku_{xx} = 0$ on $(0, \pi)$ with Neumann boundary conditions $u_x(0, t) = u_x(\pi, t) = 0$, and initial condition (1). What is the asymptotic limit $\lim_{t\to\infty} u(x, t)$?

Solution.

(a) The coefficients A_n, for $n \geq 1$ are given by

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} \phi(x)\, \cos(nx)\, dx \\ &= \frac{2}{\pi}\int_0^{\pi/2} \cos(nx)\, dx \\ &= \frac{2}{n\pi} \sin(nx)\Big|_{x=0}^{\pi} \\ &= \frac{2}{n\pi} \begin{cases} 1 & n = 4k+1, \\ 0 & n = 4k,\ n = 4k+2, \\ -1 & n = 4k+3. \end{cases} \end{aligned}$$

And $A_0 = \frac{2}{\pi}\int_0^{\pi} \phi(x)\, dx = 1$. Thus

$$\phi(x) = \frac{1}{2} + \frac{2}{\pi}\left(1\cos(x) - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \frac{1}{7}\cos(7x) + \cdots\right).$$

(b) Since the series in part (a) converges to the even extension of ϕ, which equals 1 at $x = 0$, we can evaluate by setting $x = 0$ to get

$$1 = \frac{1}{2} + \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right),$$

or

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

(c) The eigenfunctions $\cos(nx)$ are the correct ones for Neumann boundary conditions, so the solution to the heat equation with initial condition ϕ is

$$u(x, t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-kn^2 t} \cos(nx).$$

In the limit as $t \to \infty$, all the exponential terms vanish, so $\lim_{t\to\infty} u(x, t) = \frac{1}{2}A_0 = \frac{1}{2}$.

□

Problem 6. Let u solve the heat equation on $(0,1)$ with Neumann boundary conditions:

$$u_t - k u_{xx} = 0, \qquad u_x(0,t) = u_x(1,t) = 0,$$

and initial condition

$$u(x,0) = x(1-x).$$

Can there exist a point where $u_x(x,t) > 1$? Justify your answer.

Solution. We observe that $v = u_x$ also satisfies the heat equation, with *Dirichlet* boundary conditions $v(0,t) = v(1,t) = 0$, and initial condition

$$v(x,0) = \frac{d}{dx}\big(x(1-x)\big) = 1 - 2x.$$

By the strong maximum principle, the maximum value of v occurs along the initial condition or the boundaries (which are 0), and in this case that maximum is $v(0,0) = 1$, so there cannot be any points where $v(x,t) > 1$. $\square$