PDE Exam 1 Problems, Spring 2020

Problem 1. Consider the first order PDE

$$\frac{\partial u}{\partial t} - t\frac{\partial u}{\partial x} = u$$

for a function $u(t,x)$ defined on the positive quadrant $t \geq 0$, $x \geq 0$. Suppose that we wish to prescribe both an initial condition

$$u(0,x) = g(x), \quad x \geq 0$$

as well as a boundary condition

$$u(t,0) = h(t), \quad t \geq 0.$$

Show that these cannot be prescribed independently, and determine a necessary relation that must be satisfied between g and h. (Hint: is u determined by either g or h alone?)

Problem 2. Consider the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0$$

for $t \geq 0$ and $\boldsymbol{x}$ valued in a domain $\Omega \subset \mathbb{R}^n$, with the Dirichlet boundary condition

$$u(t,\boldsymbol{x}) = 0 \quad \text{for } \boldsymbol{x} \in \partial\Omega.$$

Take u to be real valued.

(a) For a solution u, show that the quantity

$$\mathcal{E}(t) = \int_\Omega \big(u(t,\boldsymbol{x})\big)^2\, d\boldsymbol{x}$$

is a non-negative and monotone decreasing function of t: i.e., $\mathcal{E}(t) \geq 0$ and $\mathcal{E}(s) \leq \mathcal{E}(t)$ whenever $s \geq t$. (Hint: consider the derivative of $\mathcal{E}(t)$.)

(b) Use part (a) to prove uniqueness for solutions to the inhomogeneous problem

$$\frac{\partial u}{\partial t} - \Delta u = f, \quad u(t,\boldsymbol{x}) = 0 \quad \text{for } \boldsymbol{x} \in \partial\Omega,$$

where $f = f(t,\boldsymbol{x})$ is an arbitrary function.

Problem 3. Find an explicit solution to the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 1, \quad t \geq 0,\ x \geq 0$$

subject to the initial and boundary conditions

$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0, \quad u(t,0) = 0.$$

Problem 4. Let $u(t, \boldsymbol{x})$ model an acoustic wave solving the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$$

in 3D (so $(t, \boldsymbol{x}) \in [0, \infty) \times \mathbb{R}^3$), and suppose the sound source is localized, so the initial data $u(0, \boldsymbol{x})$ and $\partial_t u(0, \boldsymbol{x})$ are identically 0 outside of the unit ball $B = \{x^2 + y^2 + z^2 < 1\}$. For an observer at the point $\boldsymbol{x}_0 = (3, 4, 0)$, over what time interval will the sound be heard, i.e., what is the range of t values for which $u(t, \boldsymbol{x}_0)$ may be nonzero?

Problem 5. Determine the set of eigenvalues and eigenfunctions

$$\Delta\phi = -\lambda\phi$$

for the Laplacian on a 2 dimensional rectangle $\Omega = [0, \ell_1] \times [0, \ell_2] \subset \mathbb{R}^2$ with Neumann boundary conditions

$$\frac{\partial\phi}{\partial\nu} = \nabla\phi \cdot \nu = 0 \text{ on } \partial\Omega$$

where ν denotes the outward unit normal.

Problem 6. Let Ω be the cylinder

$$\Omega = \left\{(x, y, z) : x^2 + y^2 \leq 1,\ 0 \leq z \leq \ell\right\} \subset \mathbb{R}^3$$

of height ℓ with a unit circle cross section. Use separation of variables in cylindrical coordinates (r, θ, z) to find solutions to Laplace's equation with Dirichlet boundary condition

$$\Delta u = 0, \quad u = 0 \text{ on } \partial\Omega$$

of the form $u(r, \theta, z) = h(r)w(\theta)f(z)$. Recall that cylindrical coordinates are defined by

$$(x,\ y,\ z) = (r\cos\theta,\ r\sin\theta,\ z),$$
$$(r,\ \theta,\ z) = (\sqrt{x^2 + y^2},\ \tan^{-1}(y/x),\ z).$$