Cyclic monopole chains

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23rd June 2017
Sen's conjecture and beyond, London

Monopoles on $\mathbb{R}^3$

- fix trivial bundle $E = \mathbb{R}^3 \times \mathbb{C}^2 \to \mathbb{R}^3$
- $\phi(x)$ is an $\mathfrak{su}(2)$-valued function
- $A(x)$ is an $\mathfrak{su}(2)$ connection
- $F = \mathrm{d}A + A \wedge A$ is its curvature.

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Charge $k \in \mathbb{N}$ = number zeros of ϕ.

Platonic monopoles

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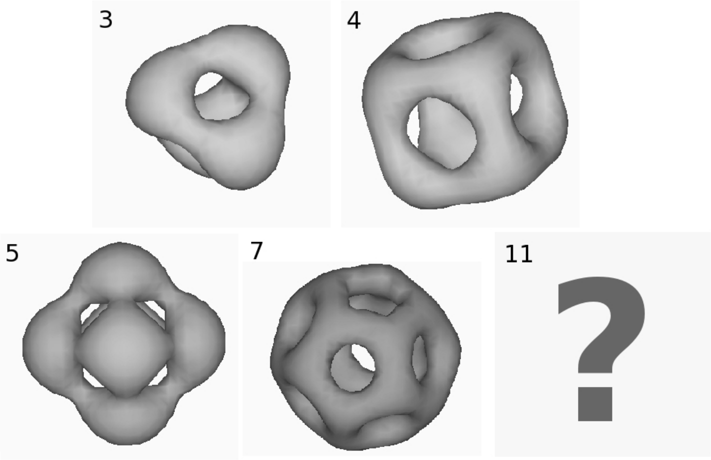

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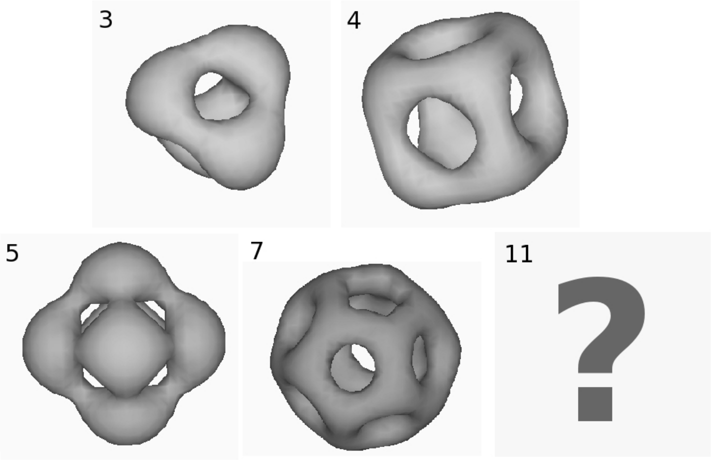

Bag conjecture. Cherry bags vs strawberry bags.

Points in “middle” of moduli space.

Monopole chains

Cherkis & Kapustin ’01, Maldonado & Ward ’13–’14, Foscolo ’16

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Boundary condition:

$$A \sim \mathrm{i}\left(b + \frac{k}{\beta}\arg\zeta\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathrm{d}\chi$$

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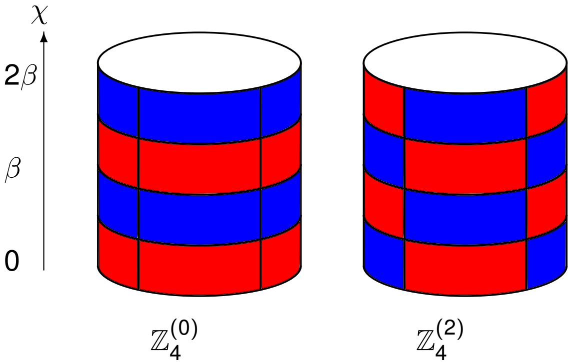
Why $\mathbb{R}^2 \times S^1$?

- Natural for Nahm transforms ($\mathbb{R}^4/\Lambda \leftrightarrow (\mathbb{R}^4)^*/\Lambda^*$)
- Interesting (ALG) hyperkähler metrics on moduli spaces
- Natural symmetry groups…

Cyclic symmetry

$\mathbb{Z}_m^{(n)}$ generated by

$$(\zeta, \chi) \mapsto (e^{2\pi \mathrm{i}/m}\zeta, \chi + n\beta/m).$$



Do there exist monopole chains with symmetry groups $\mathbb{Z}_m^{(n)}$?

Cyclic symmetry

Correspondences for monopole chains

$$\begin{tikzcd}
& \begin{array}{c}\text{Spectral curves } S \subset \mathbb{CP}^1_\zeta \times \mathbb{CP}^1_w \\ \text{+ parabolic line bundles}\end{array} & \\
\begin{array}{c}\text{Monopole chains on}\\ \mathbb{R}^2 \times S^1 \supset \mathbb{CP}^1_\zeta \setminus \{\infty\}\end{array} \arrow[ur] \arrow[rr, leftrightarrow] & & \begin{array}{c}\text{Hitchin equations on}\\ \mathbb{R} \times S^1 = \mathbb{CP}^1_w \setminus \{0, \infty\}\end{array} \arrow[ul]
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Interlude on parabolic bundles

Parabolic structures at points p encode holonomy/metric singularities

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Example (Parabolic structure at $0 \in \mathbb{C}$)

In a *unitary* gauge, consider connection

$$\begin{aligned} A &= \mathrm{i}\,\mathrm{diag}(\alpha_j)\mathrm{d}\theta \\ &= \frac{1}{2}\mathrm{diag}(\alpha_j)\left(\frac{\mathrm{d}z}{z} - \frac{\mathrm{d}\bar{z}}{\bar{z}}\right), \quad 0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k < 1. \end{aligned}$$

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(The singular metric a flag structure on the fibre at p).

Monopole chain $\to$ spectral data

- Restriction $\mathcal{E}$ of E to slice $\mathbb{C} = \mathbb{R}^2 \subset \mathbb{R}^2 \times S^1$ has holomorphic structure and parabolic structure at ∞.

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for some monic degree k polynomial P.

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- L inherits parabolic structure with weights $\alpha_\pm$ at $p_\pm$.
- $(S, L, \alpha_-, \alpha_+)$ are well-defined up to
$(L, \alpha^+, \alpha^-) \sim (L \otimes [m^+ p^+ + m^- p^-], \alpha^+ - m^+, \alpha^- - m^-)$, $m^\pm \in \mathbb{Z}$

Hitchin's equations on $\mathbb{R} \times S^1_{2\pi/\beta}$

$\hat{\phi}$ a $\mathfrak{gl}(k, \mathbb{C})$-valued function; $\hat{A}$ a $\mathfrak{u}(k)$ gauge field.

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Theorem (Cherkis-Kapustin '01)

There is a bijection between charge k monopole chains and solutions of Hitchin's equations,

$$\bar{\partial}^A \hat{\phi} = 0 \tag{HE1}$$

$$* \hat{F} - \frac{\mathrm{i}}{2}[\hat{\phi}, \hat{\phi}^\dagger] = 0, \tag{HE2}$$

satisfying the following b.c.s in local unitary gauges as $r \to \pm\infty$.

$$\hat{\phi} \sim \exp(\beta(\pm s - \mathfrak{v})/k) diag(1, e^{2\pi\mathrm{i}/k}, \dots, e^{(k-1)\pi\mathrm{i}/k}) \tag{HBC1}$$

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Solutions of (HE1), (HBC1) are called
twisted parabolic Higgs bundles. . . .

Hitchin equation $\to$ Higgs bundles

Over $\mathbb{CP}^1_w$ we have:

- A holomorphic rank k v.b. $\hat{\mathcal{E}} \to \mathbb{CP}^1_w$
- Parabolic structures at $w = 0, \infty$
- A holomorphic section ϕ of $\operatorname{End}\hat{\mathcal{E}}$ with simple poles at $w = 0, \infty$, whose residues are compatible with the parabolic structures.

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A correspondence between solutions of Hitchin's eqs and Higgs bundles is called a *Hitchin-Kobayashi correspondence*. Hitchin-Kobayashi correspondences established for both *twisted* and *parabolic* Higgs bundles, but not (yet) for *twisted parabolic* Higgs bundles.

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However, $\hat{\alpha}_\pm \neq \alpha_\pm$ because

Proposition (H)

$$\textit{par deg}\, L := \textit{deg}\, L + \alpha_+ + \alpha_- = -k$$

$$\textit{par deg}\, \hat{L} := \textit{deg}\, \hat{L} + \hat{\alpha}_+ + \hat{\alpha}_- = -k + 1.$$

Symmetric spectral data

Largest possible cyclic symmetry group is $\mathbb{Z}_{2k}$:

$$S : w - e^{\beta\mathfrak{v}}\zeta^k + 1/w = 0$$

is invariant under $(\zeta, w) \mapsto (e^{\pi \mathrm{i}/k}\zeta, -w)$.

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Action of $\mathbb{Z}_{2k}^{(m)}$:

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Proof.

Use Abel-Jacobi map: $\{L\} = \mathbb{R}^{2(k-1)}/\Lambda$. $\square$

Assuming HK correspondence, this proves existence of monopole chains with cyclic symmetry.

Symmetric Higgs bundle

E.g. $k = 3$: $\phi = \begin{pmatrix} 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \\ \nu_1 & 0 & \end{pmatrix}$. $\nu_1 \nu_2 \nu_3 = e^{-\beta s} + e^{\beta s}$.

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$\mathbb{Z}_6^{(0)}$:

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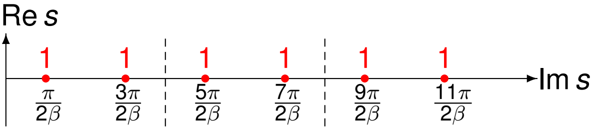

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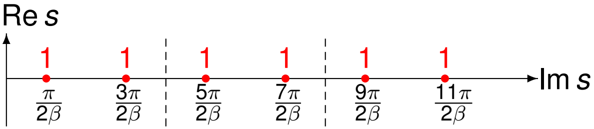

$\mathbb{Z}_6^{(2)}$: periodic up to gauge transformation.

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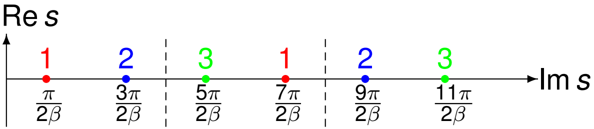

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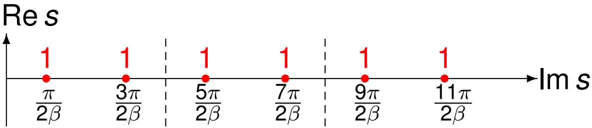

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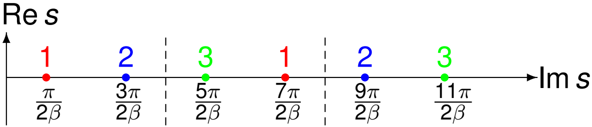
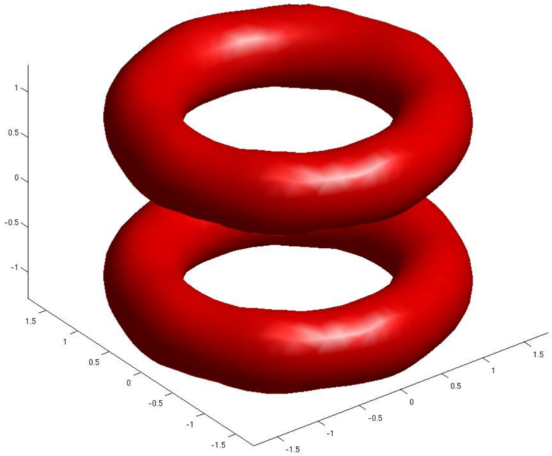
$\nu_2 = e^{-\frac{\beta s}{3}} + e^{\frac{\beta s - 2\pi\mathrm{i}}{3}}$
$\nu_3 = e^{-\frac{\beta s}{3}} + e^{\frac{\beta s - 4\pi\mathrm{i}}{3}}$

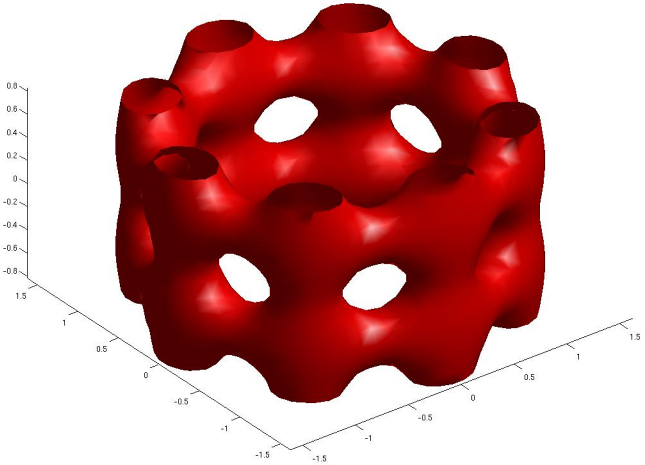


$h = \mathrm{diag}(e^{\psi_i})$ with $\triangle\psi_i = |\nu_i|^2 e^{\psi_i - \psi_{i+1}} - |\nu_{i-1}|^2 e^{\psi_{i-1} - \psi_i}$.

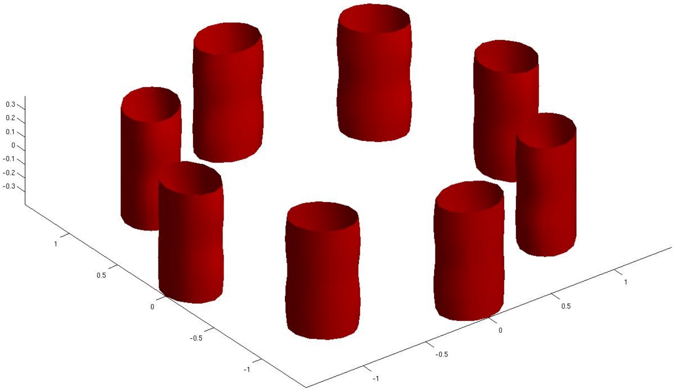
$k = 4, \quad \mathbb{Z}_8^{(0)}, \quad \beta/2\pi = 0.28$



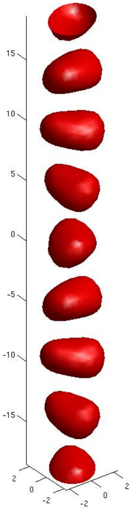
$k = 4, \quad \mathbb{Z}_8^{(0)}, \quad \beta/2\pi = 0.14$



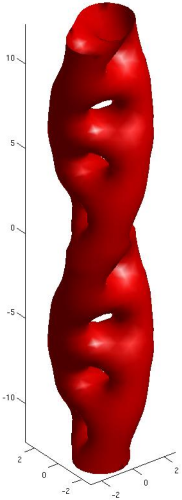
$k = 4, \quad \mathbb{Z}_8^{(0)}, \quad \beta/2\pi = 0.007$



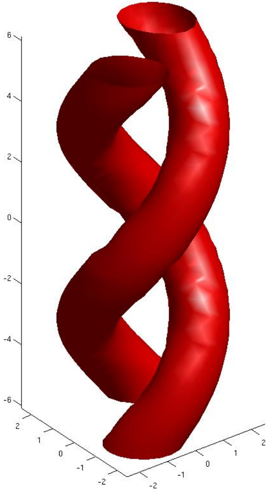
$k = 4, \quad \mathbb{Z}_8^{(2)}, \quad \beta/2\pi = 3.0$



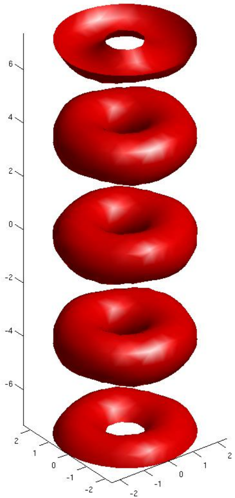
$k = 4, \quad \mathbb{Z}_8^{(2)}, \quad \beta/2\pi = 2.0$



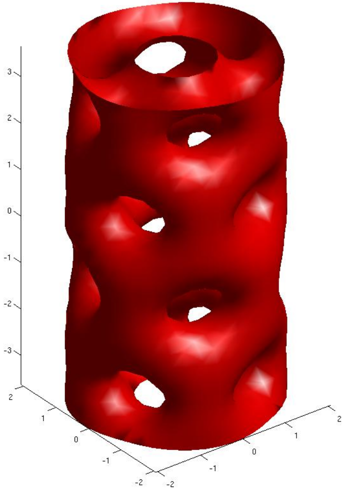
$k = 4, \quad \mathbb{Z}_8^{(2)}, \quad \beta/2\pi = 1.0$



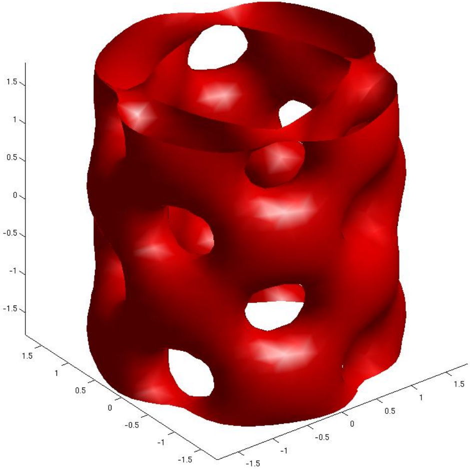
$k = 4, \quad \mathbb{Z}_8^{(4)}, \quad \beta/2\pi = 1.2$



$k = 4, \quad \mathbb{Z}_8^{(4)}, \quad \beta/2\pi = 0.6$



$k = 4, \quad \mathbb{Z}_8^{(4)}, \quad \beta/2\pi = 0.3$



Summary

- Spectral data for monopole chain has a parabolic structure.
- $\exists$ spectral data with cyclic symmetry $\mathbb{Z}_{2k}^{(2l)}$.
- $\Rightarrow \exists$ cyclic monopole chains assuming Kobayashi-Hitchin correspondence.
- Larger symmetry groups ruled out.
- Ongoing: instanton chains (calorons) with cyclic symmetry (Josh Cork).