Theorem 1. Let $M$ be a manifold with corners, and suppose $\{E_H : H \in \mathcal{M}_1(M)\}$ is a collection of vector bundles $E_H \to H$ of fixed rank over the boundary hypersurfaces such that there are given isomorphisms

$$E_H|_{H \cap H'} \cong E_{H'}|_{H \cap H'} \quad (1)$$

at the corners for each nonempty intersection $H \cap H'$. Then:

(a) There exists an open neighborhood $U \supset \partial M$ of the total boundary of $M$ and a vector bundle $E \to U$ whose restriction to each $H \in \mathcal{M}_1(M)$ is isomorphic to $E_H$.

(b) If $\{\varphi_H \in C^\infty(H; E_H)\}$ is a collection of smooth sections of the $E_H$ which are identified at corners with respect to (1), i.e.

$$\varphi_H|_{H \cap H'} \cong \varphi_{H'}|_{H \cap H'},$$

then there exists a section $\varphi \in C^\infty(U; E)$ whose restriction to each $H$ agrees with $\varphi_H$.

(c) If $\{\nabla^H\}$ is a collection of smooth connections on the $E_H$ which are identified at corners with respect to (1), then there exists a connection $\nabla$ on $E \to U$ whose restriction to each $H$ agrees with $\nabla^H$.

Remark. In particular, if the $E_H$ are trivial line bundles, part (b) gives an extension $f \in C^\infty(U)$ of a collection of smooth functions

$$\{f_H \in C^\infty(H) : H \in \mathcal{M}_1(M), f_H|_{H \cap H'} = f_{H'}|_{H \cap H'}\}.$$

The basic idea behind this result is simple. Near any boundary face, one takes $f$ to be the sum of the pullbacks from the neighboring codimension 1 boundary faces, minus the sum of the pullbacks from the codimension 2 boundary faces, plus the sum of the pullbacks from the codimension 3 faces, and so on; see (4) below.

Proof of Theorem 1.(a). For each proper $G \subset \mathcal{M}(M)$, let $E_G \to G$ be given by the restriction of $E_H$ for some $H \supset G$; this is well-defined up to isomorphism by the assumption on the $E_H$. On any product type neighborhood $V_{G,H} \cong G \times [0,1)^k$ of $G$ in $H$ (here $k$ is the codimension of $G$ in $H$) there is an isomorphism

$$(E_H)|_{V_{G,H}} \cong \pi^*_G E_G, \quad (2)$$

which follows from the smooth homotopy equivalence $V_{G,H} \sim G$.

For each $H \in \mathcal{M}_1(M)$, let $U_H \cong H \times [0,1)$ be a product type neighborhood of $H$ in $M$, and for general $G \in \mathcal{M}(M)$, let

$$\bar{E}_G = \pi^*_G E_G \to U_G, \text{ where}$$

$$U_G = \bigcap_{G \subset H \in \mathcal{M}_1(M)} U_H \cong G \times [0,1)^{\text{codim}(G)}.$$
For \( G \subset H \), it follows from (2) that there are isomorphisms \( \tilde{E}_H \cong \tilde{E}_G \) on \( U_G \cap U_H \), and therefore \( \tilde{E}_H \cong \tilde{E}_{H'} \) on \( U_H \cap U_{H'} = U_{H \cap H'} \) for any \( H, H' \in \mathcal{M}_1(M) \). It follows that the \( \tilde{E}_H \rightarrow U_H \) patch together to form a smooth bundle

\[
E \rightarrow U := \bigcup_{H \in \mathcal{M}_1(M)} U_H.
\]

For the extension of sections and connections, we first prove a local result. Let \( Y \) be a general manifold without boundary (possibly noncompact) and consider the product

\[
X = Y \times [0,1]^n.
\]

For each \( I \subset \{1, \ldots, n\} \), there is an associated boundary face \( B_I \in \mathcal{M}_{|I|}(X) \) along with an inclusion and a projection:

\[
B_I = \{ (y, x_1, \ldots, x_n) \in X : x_i = 0, \ \forall i \in I \},
\]

\[
i_I : B_I \hookrightarrow X, \ \pi_I : X \rightarrow B_I,
\]

\[
\pi_I \circ i_I = \text{Id} : B_I \rightarrow B_I.
\]

We use the notation \( B_i \) instead of \( B_{\{i\}} \) for boundary hypersurfaces. Suppose \( E \rightarrow X \) is a vector bundle, which without loss of generality (composing with an isomorphism if necessary) we may assume is of the form \( \pi_I^* E_Y = \pi_{\{1, \ldots, n\}}^* E_Y \) for a bundle \( E_Y \rightarrow Y \).

**Lemma 2.** If \( \{ \varphi_i \in C^\infty(B_i; E) : 1 \leq i \leq n \} \) is a collection of smooth sections of \( E \) on the boundary hypersurfaces which agree at all corners, i.e.

\[
(\varphi_i)|_{B_I} = (\varphi_j)|_{B_I}, \ \forall \ I \supset \{i, j\},
\]

then there is a section \( \varphi \in C^\infty(X; E) \) whose restriction to each of the \( B_i \) agrees with \( \varphi_i \).

**Proof.** For each \( I \subset \{1, \ldots, n\} \), define \( \varphi_I \in C^\infty(B_I) \) by restriction of some \( \varphi_i \) where \( i \in I \). By (3), this does not depend on the choice of \( i \in I \). Then define \( f \) by

\[
\varphi = \sum_{1 \leq |I|} (-1)^{|I|+1} \pi_I^* \varphi_I. \tag{4}
\]

Since \( E = \pi_I^* (\pi_I^* E_Y) \) for each \( I \), such an expression is well-defined. To see that \( \varphi|_{B_i} = \varphi_i \), we consider the pullback of (4) by \( i_I \), and note that

\[
\pi_I \circ i_I = \pi_{I \cup \{i\}} : B_i \rightarrow B_{I \cup \{i\}}.
\]

Thus,

\[
i_I^* \varphi = \sum_{1 \leq |I|} (-1)^{|I|+1} \pi_{I \cup \{i\}}^* \varphi_{I \cup \{i\}}
= \varphi_i + \sum_{1 \leq |I| \leq |I|} (-1)^{|I|+1} \pi_{I \cup \{i\}}^* \varphi_{I \cup \{i\}}
= \varphi_i + \sum_{1 \leq |I| \leq |I|} \left( (-1)^{|I|+1} \pi_{I \cup \{i\}}^* \varphi_{I \cup \{i\}} + (-1)^{|I \cup \{i\}|+1} \pi_{I \cup \{i\}}^* \varphi_{I \cup \{i\}} \right)
= \varphi_i. \quad \square
Proof of Theorem 1, (b) and (c). For each proper boundary face $G \in \mathcal{M}(M)$, denote by $\varphi_G \in C^\infty(M; E_G)$ the restriction to $G$ of $\varphi_H$ for some $H \in \mathcal{M}_1(M)$ such that $G \subset H$, which is independent of the choice of $H$ by the compatibility of the $\varphi_H$ at corners. Let
\[ U'_G \cong \hat{G} \times [0,1)^{\mathrm{codim}(G)} \]
be an open product-type neighborhood of the interior of $G$ (in contrast to $U_G$ defined above), set
\[ U' = \bigcup_{G \in \mathcal{M}(M)} U'_G, \]
and let $\{\chi_G\}$ be a partition of unity on $U'$ subordinate to the cover $\{U'_G\}$. Since $\hat{G} = G$ for corners of maximal codimension, it can always be arranged that $U' = U$, where $U$ is the neighborhood from part (a). Let $E \to U'$ be the bundle from (a), and regard each $\varphi_G$ is a section of $E$ over $G$.

On each $U'_G$, let $\tilde{\varphi}_G \in C^\infty(U'_G; E)$ be a section restricting to $\varphi_H$ on $U'_G \cap H$ for each hypersurface $H$ such that $G \subset H$, as in Lemma 2. Then
\[ \varphi = \sum_{G \in \mathcal{M}(M)} \chi_G \tilde{\varphi}_G \in C^\infty(U; E) \tag{5} \]
has the desired properties.

We claim that the same procedure allows the connections to be extended. Of course it does not make sense to take a general linear combination of connections since they do not form a vector space but rather an affine space modelled on (endomorphism-valued) one-forms. However, provided the coefficients in the linear combination sum identically to 1, such a linear combination in an affine space makes sense. Indeed, if $\{\nabla_1, \ldots, \nabla_n\}$ are connections on a given bundle $E \to X$, and $\{a_1, \ldots, a_n : a_i \in C^\infty(X)\}$ are functions such that $\sum_i a_i \equiv 1$, then we may take as a definition
\[ a_1 \nabla_1 + \cdots + a_n \nabla_n := \nabla_1 + a_2(\nabla_2 - \nabla_1) + \cdots + a_n(\nabla_n - \nabla_1), \tag{6} \]
where $(\nabla_1 - \nabla_1) \in \Omega^1(X; \operatorname{End}(E))$ denotes the one-form $\alpha_i$ such that $\nabla_1 + \alpha_i = \nabla_i$. Since $a_1 = 1 - \sum_{i=2}^n a_i$, (6) is well-defined independent of the ordering of the $\nabla_i$.

It then suffices to note that the coefficients in (4) sum to the identity, since
\[ \sum_{1 \leq |I|} (-1)^{|I|+1} = \sum_{1 \leq k \leq n} \binom{n}{k} (-1)^k = 1 - (1 - 1)^n = 1, \]
and of course the coefficients in (5) sum to the identity by construction. \(\square\)

Northeastern University, Department of Mathematics
E-mail address: c.kottke@neu.edu