Math 1580 – Problem Set 9. Due Friday Nov. 18, 4pm

Updated 11/13 to fix a typo in Problem 1. Thanks to Sarah for the catch.

Problem 1. Recall the following method of *cofactor expansion* to calculate the determinant of an $n \times n$ matrix A. Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by removing the ith row and jth column, and a_{ij} denote the i, jth entry of A. Then $\det(A)$ can be calculated by fixing a row, say row k, and computing

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+k} a_{kj} \det(A_{kj})$$

Similarly, we may fix a column instead, say column k, and compute

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det(A_{ik})$$

Given A as above, define the *cofactor matrix*[1] B to be the matrix whose i, j entry is

$$b_{ij} = (-1)^{i+j} \det(A_{ji})$$

(a) Prove that

$$AB = BA = \det(A) I_n$$

where I_n is the identity matrix. Conclude that provided $\det(A) \neq 0$, A^{-1} is given by

$$A^{-1} = \frac{1}{\det(A)} B$$

(b) Use this to prove that if A has integer entries and $\det(A) = \pm 1$, then A^{-1} has these same properties.

(c) Conclude that the $n \times n$ matrices with integer entries and determinant ± 1 form a group with respect to matrix multiplication, which we call $\mathrm{GL}(n, \mathbb{Z})$.

Problem 2. Let L be a lattice in $\mathbb{R}^n$, and suppose $\dim(L) = n = \dim(\mathbb{R}^n)$. Show that a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset L$ is a basis for L if and only if

$$L \cap \mathcal{F}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = 0 \tag{1}$$

where $\mathcal{F}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is the fundamental domain for $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ defined as in class by

$$\mathcal{F}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \{t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n : 0 \leq t_i < 1, \text{ for all } i\}$$

Some hints:

(a) To show that (1) holds if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis, suppose that there is a vector $\mathbf{v} \in L \cap \mathcal{F}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and show that $\mathbf{v}$ must be the zero vector.

(b) To show the other direction, let L' be the lattice generated by the $\mathbf{v}_i$, so that $L' \subseteq L$. To show that $L \subseteq L'$, let $\mathbf{v} \in L$ and write $\mathbf{v}$ as a linear combination (not necessarily with integer entries) of the $\mathbf{v}_i$, and use this to find a vector $\mathbf{v}' \in L'$ such that $\mathbf{v} - \mathbf{v}' \in \mathcal{F}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Conclude that $\mathbf{v}$ must equal $\mathbf{v}'$.

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[1] This is sometimes also called the "adjugate matrix." Very unfortunately, it also sometimes called the "adjoint matrix," which is a terrible practice since there is a different matrix obtained from A which is also called the adjoint and deserves the title much more.

Problem 3. Let $L \subset \mathbb{R}^m$ be a lattice with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. We showed in class how to compute $\det(L)$ as $|\det(F(\mathbf{v}_1, \ldots, \mathbf{v}_n)|$ in the case that $m = n$, where

$$F(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \begin{pmatrix} | & \cdots & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & \cdots & | \end{pmatrix} \tag{2}$$

is the matrix whose columns consist of the components of the $\mathbf{v}_i$ as vectors in $\mathbb{R}^m$.

This problem will give a way to compute this quantity even when $m > n$. Note that in this case, the matrix (2) is still well-defined as a $m \times n$ matrix.

(a) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors in $\mathbb{R}^m$, define the *Gram matrix* $\mathrm{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ to be the $n \times n$ matrix whose i, j entry is the quantity

$$[\mathrm{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_n)]_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j, \quad 1 \le i, j \le n.$$

Show that

$$\mathrm{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = F(\mathbf{v}_1, \ldots, \mathbf{v}_n)^{\mathrm{T}} F(\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

(b) Show that if $n = m$, then

$$\det\big(\mathrm{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_n)\big) = \det(L)^2. \tag{3}$$

(c) Show that if $m > n$, then (3) still holds. Here are some hints:

(i) Argue that (3) holds if the $\mathbf{v}_i$ all lie in the subspace $\{(x_1, \ldots, x_m) : x_{n+1} = \cdots = x_m = 0\} \subset \mathbb{R}^m$. We will reduce to this case below.

(ii) Remind yourself (or go learn!) that a (real valued) matrix is *orthogonal* if $\det(R) = \pm 1$, and that such matrices satisfy $R\mathbf{v}_i \cdot R\mathbf{v}_j = \mathbf{v}_i \cdot \mathbf{v}_j$. Recall also that a matrix whose columns form a set of orthonormal vectors is an orthogonal matrix, and that orthogonal transformations preserve lengths, areas, volumes and so on. You may assume all these facts.

(iii) Enlarge the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ for $\mathbb{R}^m$ by adding $m - n$ additional independent vectors. Let $\{\mathbf{v}_1^*, \ldots, \mathbf{v}_m^*\}$ be an orthonormal set of vectors obtained from $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ by the Gram-Schmidt procedure. Observe that the subspace spanned by the first n vectors in $\{\mathbf{v}_i^*\}$ is the same as that spanned by our original vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

(iv) Form the orthogonal matrix R whose columns are the vectors $\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*$. Show that the linear transformation defined by R sends the subspace $\{x_{n+1} = \cdots = x_m = 0\}$ to the space spanned by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Argue that the inverse R^{-1} is also an orthogonal transformation, which does the reverse. Conclude the problem by showing that

$$\begin{aligned} &\mathrm{Gram}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \mathrm{Gram}(R^{-1}\mathbf{v}_1, \ldots, R^{-1}\mathbf{v}_n), \text{ and} \\ &\mathrm{Vol}_n\big(\mathcal{F}(\mathbf{v}_1, \ldots, \mathbf{v}_n)\big) = \mathrm{Vol}_n\big(\mathcal{F}(R^{-1}\mathbf{v}_1, \ldots, R^{-1}\mathbf{v}_n)\big) \end{aligned}$$

using your result from (cii) above.