Math 2420 – Problem Set 1, due Monday 2/6.

Updated 2/2: typo fixed in problem 4. The retraction identity should be $r \circ i = \mathrm{Id}$.

Problem 1. A proof of the Mayer-Vietoris sequence from the axioms:

(a) Given a commutative diagram of abelian groups

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & A_n & \xrightarrow{i} & A'_n & \xrightarrow{j} & A''_n & \xrightarrow{k} & A_{n-1} & \xrightarrow{i} & A'_{n-1} & \xrightarrow{j} & A''_{n-1} & \longrightarrow & \cdots \\
 & & \cong\big\downarrow e & & \big\downarrow f & & \big\downarrow g & & \cong\big\downarrow e & & \big\downarrow f & & \big\downarrow g & & \\
\cdots & \longrightarrow & B_n & \xrightarrow{i} & B'_n & \xrightarrow{j} & B''_n & \xrightarrow{k} & B_{n-1} & \xrightarrow{i} & B'_{n-1} & \xrightarrow{j} & B''_{n-1} & \longrightarrow & \cdots
\end{array}$$

in which the rows are (long) exact, and every third vertical map (e) is an isomorphism, show that there is an induced long exact sequence

$$\cdots \longrightarrow A'_n \longrightarrow A''_n \oplus B'_n \longrightarrow B''_n \longrightarrow A'_{n-1} \longrightarrow \cdots$$

(Hint: define the map $A'_n \longrightarrow A''_n \oplus B'_n$ by $\alpha \longmapsto \big(j(\alpha), f(\alpha)\big)$ and the map $A''_n \oplus B'_n \longrightarrow B''_n$ by $(\alpha, \beta) \longmapsto g(\alpha) - j(\beta)$. What should the third map be?)

(b) Use the above result to prove the Mayer-Vietoris sequence for homology using only the axioms for homology. (Hint: consider the long exact sequences of the pairs (X, A) and $(B, A \cap B)$, and use excision to relate the relative homology of these two pairs.)

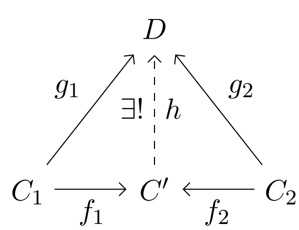
Problem 2. Assume that a space $X = \bigcup_{i \leq n} A_i$ has a covering by n open sets A_i, with the property that any intersection $A_I = \bigcap_{i \in I} A_i$, $I \subset \{1, \ldots, n\}$ of these sets is either empty or has trivial homology:

$$\widetilde{H}_*(A_I) = 0, \quad \forall\, I \subset \{1, \ldots, n\}\,.$$

Use the Mayer-Vietoris sequence to show that $\widetilde{H}_k(X) = 0$ for $k \geq n-1$. Give some examples that indicate that this is sharp, (i.e. $\widetilde{H}_k(X) \neq 0$ for some $k < n-1$.)

Problem 3. In any category $\mathcal{C}$, the *coproduct* (also called the *direct product*) of two objects C_1 and C_2 (if it exists) is an object C' with morphisms $f_i : C_i \longrightarrow C'$, $i = 1, 2$ which has the following *universal property:*

- for any other object $D \in \mathcal{C}$ with morphisms $g_i : C_i \longrightarrow D$, $i = 1, 2$, there exists a *unique* morphism $h : C' \longrightarrow D$ such that $g_i = h \circ f_i$:



(a) Show that a coproduct, if it exists, is unique up to unique isomorphism (i.e. if C' and C'' are both coproducts of C_1 and C_2 then there is a unique isomorphism $C' \longrightarrow C''$).

(b) Show that coproducts always exist in the following categories, given by the following operations:

  (i) In the category Top of topological spaces, $C' = C_1 \sqcup C_2$.

  (ii) In the category Top_* of based topological spaces, $C' = C_1 \vee C_2$.

  (iii) In the category AbGp of abelian groups, $C' = C_1 \oplus C_2$.

(In particular, the addition axioms for homology say that H_* and $\widetilde{H}_*$ take coproducts to coproducts.)

(c) Show that the operation of adjoining a disjoint basepoint,

$$(-)_+ : X \longmapsto X_+ = X \sqcup \{\mathrm{pt}\}$$

is a functor from Top to Top_* which preserves coproducts, i.e.

$$\big(X_1 \sqcup X_2\big)_+ = (X_1)_+ \vee (X_2)_+.$$

Problem 4. Split exact sequences (c.f. Hatcher p. 147):

(a) For a short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{f} A' \xrightarrow{g} A'' \longrightarrow 0 \tag{1}$$

show that the following conditions are equivalent:

  (i) There exists a homomorphism $\phi : A'' \longrightarrow A'$ such that $g \circ \phi = \mathrm{Id}$.

  (ii) There exists a homomorphism $\psi : A' \longrightarrow A$ such that $\psi \circ f = \mathrm{Id}$.

  (iii) There exists an isomorphism $A' \cong A \oplus A''$ such that $f(a) = (a, 0)$ and $g(a, b) = b$.

If these conditions hold, we say that (1) is *split*, or *split exact*.

(b) Let $i : A \subset X$ be the inclusion of a subspace, and recall that a *retraction* is a map $r : X \longrightarrow A$ such that $r \circ i = \mathrm{Id}$. Show that if there is a retraction[1] X onto A, then the long exact sequence

$$\cdots \xrightarrow{\partial} H^{k-1}(A) \longrightarrow H^k(X, A) \longrightarrow H^k(X) \longrightarrow H^k(A) \xrightarrow{\partial} \cdots$$

degenerates into split short exact sequences, and

$$H^k(X) \cong H^k(X, A) \oplus H^k(A), \quad \forall\, k$$

[1] it need not be a *deformation retraction*, which in addition satisfies $i \circ r \simeq \mathrm{Id}$.