Math 350 Problem Set 6 Solutions to Part I

Part I

1. **Additivity over general regions**. Call a region $A \subset \mathbb{R}^2$ "nice" if its boundary consists of a piecewise set of differentiable curves. Let A and B be nice regions, with $B \subset A$. Let C be the set of points which are in A but not in B:
$$C = A \setminus B = \{(x, y) \mid (x, y) \in A, (x, y) \notin B\}$$

   (a) (8pts) Show that, if f is continuous (or even bounded, with a nice set of discontinuities), then
$$\iint_A f\,dA = \iint_B f\,dA + \iint_C f\,dA.$$
(Hint: Recall how $\iint_A f\,dA$ is defined: $\iint_A f\,dA = \iint_R f^*\,dA$ for an auxilliary function f^* such that $f^* = f$ on A and 0 otherwise. If f^* is such a function for A, and f^{**} is such a function for B, what is $f^* - f^{**}$?)

   (b) (7pts) Iterate this to prove that, if $A = \bigcup_{i=1}^N A_i$ is a decomposition of a nice set A into N pieces which are also nice, then
$$\iint_A f\,dA = \sum_{i=1}^N \iint_{A_i} f\,dA.$$

Let $R \subset \mathbb{R}^2$ be a rectangle containing A (and hence also containing B and C). By definition,
$$\iint_B f\,dA = \iint_R f_B\,dA, \quad \text{and} \quad \iint_C f\,dA = \iint_R f_C\,dA$$
where
$$f_B(x, y) = \begin{cases} f(x, y) & (x, y) \in B, \\ 0 & \text{otherwise,} \end{cases}$$
and
$$f_C(x, y) = \begin{cases} f(x, y) & (x, y) \in C, \\ 0 & \text{otherwise.} \end{cases}$$
Since $(x, y) \in A$ means that either $(x, y) \in B$ or $(x, y) \in C$, but not both, we have
$$f_B(x, y) + f_C(x, y) = f_A(x, y) = \begin{cases} f(x, y) & (x, y) \in A \\ 0 & \text{otherwise.} \end{cases}$$
By linearity of the integral,
$$\iint_A f\,dA = \iint_R f_A\,dA = \iint_R (f_B + f_C)\,dA = \iint_R f_B\,dA + \iint_R f_C\,dA = \iint_B f\,dA + \iint_C f\,dA.$$
and we're done.

Now if $A = \bigcup_{i=1}^N A_i$ is a decomposition into disjoint sets, we have
$$\begin{aligned}\iint_A f\,dA &= \iint_{A_1} f\,dA + \iint_{A_2 \cup \cdots \cup A_N} f\,dA \\ &= \iint_{A_1} f\,dA + \iint_{A_2} f\,dA + \iint_{A_3 \cup \cdots \cup A_N} f\,dA = \cdots \\ &= \sum_{i=1}^N \iint_{A_i} f\,dA.\end{aligned}$$

2. **Generalized cones**. Let R be a nice region of $\mathbb{R}^2$. The cone over R, with height h, is the set of all points between $(0,0,h)$ and $(x,y,0)$ where $(x,y) \in R$:

$$C_h(R) = \{t(0,0,h) + (1-t)(x,y,0) \mid (x,y) \in R, 0 \le t \le 1\}.$$

(a) (3pts) Draw a sketch of $C_h(R)$, so you can see what's going on.

(b) (12pts) By setting up and evaluating a triple integral, prove that

$$\text{Vol}(C_h(R)) = \frac{1}{3}h\text{Area}(R)$$

(Hints: Think about the cross sections $z = \text{const}$. How do lengths in these cross sections scale with z? How does the area of the cross section scale with z? Your integral will probably be an iterated integral in dz and dA.)

The area of R is given by

$$\iint_R dA,$$

and if we scale all lengths in R by a factor a, the new area will be

$$a^2 \iint_R dA.$$

For each $z \in [0,h]$, the cross section of $C_h(R)$ is a copy of R, with all lengths scaled by a factor of $(h-z)/h$. Thus the volume of $C_h(R)$ is

$$\int_0^h \frac{(h-z)^2}{h^2} \iint_R dA\, dz = \iint_R \int_0^h \frac{(h-z)^2}{h^2}\, dz\, dA = \iint_R \left(\frac{-(h-z)^3}{3h^2}\right)_{z=0}^{z=h} dA = \iint_R \frac{h}{3}\, dA = \frac{1}{3}h\text{Area}(R).$$