Math 350 Problem Set 9 (due Friday 11/19 by 3pm)

Part I

1. (5pts) Prove that the curl of a gradient is always zero, and that the divergence of a curl is always zero (assuming f or $\mathbf{F}$ are C^2 functions):

$$\nabla \times (\nabla f) = 0, \quad \text{and} \quad \nabla \cdot (\nabla \times \mathbf{F}) = 0 \qquad \text{for all } f \text{ and } \mathbf{F}.$$

   The divergence of a gradient is not necessarily zero, however. The **Laplacian** is defined to be the operator Δ which takes a scalar function to a scalar function by the formula

$$\Delta f = \nabla \cdot (\nabla f).$$

   (Some people (including our textbook, and many physics texts) use the notation ∇^2 instead of Δ for the Laplacian.)

   Write down the expression for Δf in 1, 2 and 3 dimensions.

2. (10pts) Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ be a scalar function. Use Green's Theorem to prove the formula

$$\iint_R f\Delta f + \nabla f \cdot \nabla f \, dA = \oint_{\partial R} f \nabla f \cdot \hat{\mathbf{n}} \, ds$$

3. (15pts) **Laplace's Equation** is the (partial differential) equation $\Delta u = 0$, for a scalar function $u(x, y)$. Functions u which satisfy Laplace's equation (called **harmonic functions**) are deeply important in mathematics and physics. A typical problem that arises is to find a function u such that $\Delta u = 0$ on the interior of a region $R \subset \mathbb{R}^2$, and such that u is equal to some fixed function at the boundary of R, i.e. that $u(x, y) = g(x, y)$ for all $(x, y) \in \partial R$. Such a u is said to be a solution of the **boundary value problem**

$$\begin{cases} \Delta u(x, y) = 0 & (x, y) \in R \\ u(x, y) = g(x, y) & (x, y) \in \partial R \end{cases} \tag{1}$$

   and g is called the **boundary value.**

   This is a physical model for the following situation. Take the curve in $\mathbb{R}^3$ given by the graph of g over ∂R; that is, the set $\{(x, y, g(x, y)) \in \mathbb{R}^3 \mid (x, y) \in \partial R\}$. Picture this curve in space as a rigid wire, and then dip this wire into a solution of soap and water, or imagine attaching a rubber sheet to it. The soap film or rubber sheet, stretched out in the space inside the wire ring, defines a surface which is a solution to (1).

   For mathematical purposes, it is nice to know two things about such differential equation problems: A) that solutions *exist* (i.e. given a g, that at least one function u exists which solves (1), and B) that solutions are *unique* (so that there is only one solution for each choice of g). These questions can be very difficult to answer for general partial differential equations – in fact, much ongoing mathematical research today is concerned with existence and uniqueness for various partial differential equations (PDE).

   For Laplace's equation (1) however, one can prove uniqueness using the formula from problem 2. (Existence is harder, and somewhat beyond the scope of this class.) So the problem is to show that if u_1 and u_2 are solutions to (1), then $u_1 = u_2$. Here are some steps.

   (a) Suppose g is fixed, and u_1, u_2 are two solutions to (1). Show that $u_1 - u_2$ is a solution to Laplace's Equation, but with a different boundary value. What is the boundary value of $u_1 - u_2$?

(b) Set $f = u_1 - u_2$. Use the formula from problem 2 to show that

$$\iint_R \nabla f \cdot \nabla f \, dA = \iint_R \|\nabla f\|^2 \, dA = 0.$$

(You will need to use the fact that f is harmonic, as well as the particular boundary value of f.)

(c) Conclude that we must have

$$\nabla f = 0.$$

(Remember that we showed a while back that if $\iint_R h \, dA = 0$ for a nonnegative, continuous function h, then $h = 0$).

(d) Thus $f = u_1 - u_2$ must be a constant in R. Argue that this constant must be 0 because of the boundary value of f. Therefore,

$$u_1 = u_2$$

and we have proved that solutions to (1) are in fact unique.

Part II

1. Let $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$, and suppose $\mathcal{C}$ is a circle of radius 1 with center at $(1, 0)$, oriented clockwise. Compute

$$I = \oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} \, ds$$

two ways:

(a) (5pts) Directly.

(b) (5pts) Using Green's Theorem.

2. Let $\mathbf{F}(x, y) = xy^2\mathbf{i} + xy\mathbf{j}$, and suppose $\mathcal{C}$ consists of the straight line segments from $(0, 0)$ to $(1, 0)$ to $(0, 1)$ and back to $(0, 0)$, oriented counterclockwise. Compute

$$I = \oint_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{n}} \, ds$$

two ways:

(a) (5pts) Directly.

(b) (5pts) Using Green's Theorem.

3. (10pts) Find the closed curve $\mathcal{C}$ in $\mathbb{R}^2$ which *maximizes* the line integral

$$\oint_{\mathcal{C}} (x^2 - 2)y \, dx - (y^2 - 2)x \, dy.$$

That is, find the curve over which this integral has the largest possible value.

(Hint: Green's Theorem.)