Summary of Current Research

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Introduction

The mathematical questions I find most interesting are those concerning moduli spaces and topological invariants in geometry—especially involving noncompact, singular or infinite dimensional spaces—which can be approached through the analysis of PDE. I specialize in methods of geometric microlocal analysis and index theory, and I have a keen interest in problems arising from mathematical physics, particularly gauge theory and string theory. Here I give an overview of three of my current research topics.

- The first is the study of the moduli spaces of magnetic monopoles, both on $\mathbb{R}^3$ and more general 3-manifolds. A particular goal is to construct a compactification of the moduli space for monopoles on $\mathbb{R}^3$ with nice metric asymptotics, which should lead to a proof of Sen's conjecture for the L^2 cohomology of the moduli space.
- The second is the rigorous development of Dirac operators on the free loop space of a compact manifold, with the goal of understanding Witten's index formula for the Dirac-Ramond operator, whose index computes the genus associated to topological modular forms.
- The last concerns the categories of manifolds with corners and so-called generalized corners, and in particular the development of a systematic treatment of blow-up in these categories, using ideas from toric and logarithmic algebraic geometry.

1 Magnetic Monopoles

The moduli space, $\mathcal{M}_k$, of charge k, SU(2) magnetic monopoles on $\mathbb{R}^3$ is an interesting space from a variety of directions. In addition to being a moduli space in 3-dimensional gauge theory, $\mathcal{M}_k$ has several equivalent characterizations, for instance as the space of based, degree k rational maps on $\mathbb{C}P^1$ [Don84], among others [Hit82, Nah82, Hit83]. From a PDE perspective, $\mathcal{M}_k$ comprises the stable manifold of static solutions to a system of nonlinear dispersive equations [Man82]. Monopoles are examples of *solitons*: though defined in terms of fields on $\mathbb{R}^3$, they nevertheless have particle-like characteristics, such as mass, center of mass, and charge. They behave asymptotically like point magnetic charges in Maxwellian electromagnetism, but have more complicated structure coming from fact that the structure group, SU(2), is nonabelian, in contrast to the abelian group U(1) of classical electromagnetism. Moreover, while the intuitive picture of a charge k monopole as a superposition of individual monopoles (of unit charge) is approximately valid at large separation distance, it breaks down—in typical soliton fashion—as the constituent monopoles become close together and lose their individual identities.

Compactification of $\mathcal{M}_k$ and metric asymptotics. Much is known about the space $\mathcal{M}_k$: it is a smooth, noncompact manifold of dimension $4k$ [JT80, Tau83, Tau85]; moreover it carries a natural hyperKähler metric [HKLR87] with respect to which it is a complete Riemannian manifold. Its absolute and relative cohomology were computed in [SS96], but its L^2 cohomology—the dimensions of the spaces of L^2 harmonic forms—was conjectured in [Sen94] based on predictions from string theory, but remains unproved. My ongoing project with Michael Singer, Richard Melrose and Karsten Fritzsch is to construct a compactification of $\mathcal{M}_k$ as a manifold with corners (or possibly generalized corners, see §3) on which the metric has a complete asymptotic expansion, after which the L^2 cohomology will be computable in terms of the rate of decay of harmonic forms.

In more detail, monopoles are equivalence classes of solutions to the Bogomolny equation

$$\mathcal{B}(A,\Phi) = d_A\Phi - \star F_A = 0, \tag{1.1}$$

where F_A is the curvature of a connection, A, on a principal SU(2) bundle $P \longrightarrow \mathbb{R}^3$; Φ is a section of the associated Lie algebra bundle $\operatorname{ad} P$; d_A is the covariant exterior derivative; and $\star$ denotes the Hodge star operator. Solutions are considered which have a suitably prescribed boundary value (A_0, Φ_0) on the 2-sphere at infinity (meaning the boundary $\partial\overline{\mathbb{R}^3} = \mathbb{S}^2$ of the radial compactification of $\mathbb{R}^3$), and two solutions are considered equivalent if they are intertwined by an element of the gauge group $\mathcal{G}_0 = \big\{g \in \operatorname{Aut}(P) : g|\partial\overline{\mathbb{R}^3} = 1\big\}$. The map $\Phi_0 : \mathbb{S}^2 \longrightarrow \mathfrak{su}(2) \cong \mathbb{R}^3$ has fixed norm pointwise, and the homotopy class $k = [\Phi_0] \in \pi_2(\mathbb{S}^2) = \mathbb{Z}$ is an invariant known as the *charge* of the monopole. The space of such equivalence classes is the moduli space $\mathcal{M}_k$.

The tangent space, $T_{(A,\Phi)}\mathcal{M}_k$, to a monopole is naturally identified with the L^2 nullspace of an elliptic differential operator on sections of the bundle $(T^* \oplus \mathbb{R}) \otimes \operatorname{ad} P$, and for $(a,\phi) \in T_{(A,\Phi)}\mathcal{M}_k$, the L^2 pairing $g_k(a,\phi) = \int_{\mathbb{R}^3} |a|^2 + |\phi|^2$ determines the Riemannian metric on $\mathcal{M}_k$.

The noncompactness of $\mathcal{M}_k$ has to do with the soliton characteristics of monopoles—the noncompact part of the moduli space is approximated by superpositions of lower charge monopoles, which may go off to infinity along different directions in $\mathbb{R}^3$, as shown in [AH88]. To construct a compactification, we define *ideal monopoles* as data which represent the limiting configurations of lower charge monopoles 'at infinity' and show that these can be deformed to nearby monopoles in smooth families.

In the first phase of the project we construct the boundary faces of codimension 1 of the compactification; these are indexed by partitions $k = k_0 + k_1 + \cdots + k_N$, where $N \geq 1$ and $k_i > 0$ for $i \geq 1$. Geometrically, these boundary hypersurfaces are parameterized by ideal monopoles whose moduli space, $\mathcal{I}_{\underline{k}}$, $\underline{k} = (k_0, \ldots, k_N)$ is a nontrivial fiber bundle

$$\mathcal{I}_{\underline{k}} \longrightarrow \mathcal{C}_N^* = \Big\{(\zeta_1, \ldots, \zeta_N) \in (\mathbb{R}^3)^N : \zeta_i \neq \zeta_j \neq 0,\ \sum |\zeta_i|^2 = 1\Big\}$$

over the space of distinct, non-zero configurations of N points in $\mathbb{R}^3$ up to scaling, with fiber

$$(\mathcal{I}_{\underline{k}})_{\underline{\zeta}} \cong \mathcal{M}_{k_0} \times \mathcal{M}^0_{k_1} \times \cdots \times \mathcal{M}^0_{k_N}$$

consisting of the spaces of N framed, centered monopoles of charges adding up to $k - k_0$, along with the space of uncentered framed monopoles of charge k_0. Here $\mathcal{M}^0_k$ is the moduli space of monopoles with center of mass at 0, or equivalently the quotient, $\mathcal{M}_k/\mathbb{R}^3$, with respect to the free action by translations. These ideal monopoles represent the decomposition of a charge k monopole into N lower charge 'monopole clusters' which have gone off to infinity along paths $z_i = \zeta_i/\varepsilon$, where $\varepsilon \to 0$, with an additional charge k_0 monopole 'left behind' in the interior $\mathbb{R}^3$.

Theorem 1.1 ([KS15])**.** *Fix a partition $k = k_0 + k_1 + \cdots k_N$ and an ideal monopole $\iota_0 \in \mathcal{I}_{\underline{k}}$. There exists a neighborhood $\mathcal{U} \ni \iota_0$, a constant $\varepsilon_0 > 0$ and a smooth map*

$$\Psi : \mathcal{U} \times (0, \varepsilon_0) \hookrightarrow \mathcal{M}_k(\mathbb{R}^3)$$

which is a local diffeomorphism onto its image, such that $\Psi(\iota,\varepsilon) \to \iota$ as $\varepsilon \to 0$, with a complete asymptotic expansion in ε, and the hyperKähler metric splits as a Riemannian product to first order:

$$\Psi^*(g_k) \sim g_{k_0} \oplus \big(\bigoplus_{i=1}^N g_{k_i} \oplus 2\pi k_i \eta\big) + \mathcal{O}(\varepsilon)$$

with respect to a canonical splitting

$$T\big(\mathcal{I}_{\underline{k}} \times (0,\infty)\big) \cong T\mathcal{M}_{k_0} \oplus \big(\bigoplus_{i=1}^{N} T\mathcal{M}^0_{k_i} \oplus \mathbb{R}^3\big).$$

Here we have identified the base $\mathcal{C}^*_N \times (0,\infty)$ with the space $\mathcal{C}_N \subset (\mathbb{R}^3)^N$ of configurations (not modulo scaling) via $(\zeta_1,\ldots,\zeta_N,\varepsilon) \longmapsto (\zeta_1/\varepsilon,\ldots,\zeta_N/\varepsilon)$, η is the standard Euclidean metric on $\mathbb{R}^3$, and g_{k_i} is the L^2 metric on the moduli space $\mathcal{M}_{k_i}$ (possibly restricted to the sub-moduli space of centered monopoles).

In fact, we obtain a complete asymptotic expansion of the metric $\Psi^*(g_k)$ in ε, and we can explicitly compute the subleading order term as well; the result generalizes the asymptotic metric of Gibbons and Manton in [GM95] for the case $k_i = 1$, $i = 1,\ldots,N$, $k_0 = 0$.

Future work. The next step in the project is to construct the higher codimension faces of the moduli space. These faces represent configurations of clusters which go off to infinity at different rates which are not uniformly comparable. The rates may be arranged in a hierarchy: the leading order rate corresponds to a decomposition of the monopole into clusters of charge $k_1,\ldots,k_N$; at the next order the k_i clusters decompose into further subclusters of charge $k_{i1},\ldots,k_{iN_i}$; and so on. The combinatorics and geometry (involving the inductive compactifications of the moduli spaces $\mathcal{I}_{\underline{k}}$ and how these fit together) is understood, and the analysis required for the gluing is only a slightly more complicated variation on that involved in Theorem 1.1.

Problem 1.2. *Construct all higher codimension corners for the compactification of $\mathcal{M}_k$, and show that together these classify all deformations of (ideal) monopoles. Show that the resulting universal deformation space, which is a manifold with (generalized) corners containing $\mathcal{M}_k$, is compact. Finally, derive from the asymptotic expansions of the metric the rate of decay of harmonic forms at the various boundary faces, and use this to prove Sen's conjecture.*

Monopoles on other manifolds. In addition to $\mathbb{R}^3$, monopoles have historically been studied on hyperbolic 3-space [Ati84, MS96, MS00], on manifolds with (conformally compact) asymptotically hyperbolic ends [Bra89], and to a limited extent in [Flo95b, Flo95a] on manifolds with Euclidean ends (i.e., isometric to $\mathbb{R}^3$ outside a compact set).

I am interested in monopoles on asymptotically conic 3-manifolds. In [Kot15c], I computed the virtual dimensions of the framed and unframed moduli spaces of monopoles in this setting in terms of the *deformation complex*

$$0 \longrightarrow T_{\mathrm{Id}}\mathcal{G} \xrightarrow{D_0} T_{(A,\Phi)}\mathcal{C} \xrightarrow{D_1} C^\infty(X; T^*X \otimes \mathrm{ad}\,P) \longrightarrow 0. \tag{1.2}$$

Here $\mathcal{C}$ is the configuration space of pairs (A,Φ) on an $\mathrm{SU}(2)$ bundle $P \longrightarrow X$, D_0 is the differential of the action of the gauge group $\mathcal{G}$ on $\mathcal{C}$, and D_1 is the linearization of (1.1). The tangent space to the monopole moduli space at a smooth point (A,Φ) is the quotient $\mathrm{Ker}\,D_1/\,\mathrm{Im}\,D_0$, while the kernel of D_0 and cokernel of D_1 represent obstructions to smoothness. The *virtual dimension* of the moduli space is the Euler characteristic $\dim(\mathrm{Ker}\,D_1/\,\mathrm{Im}\,D_0) - \dim(\mathrm{Ker}\,D_0) - \dim(\mathrm{Coker}\,D_1)$; it coincides with the true dimension where the space is smooth.

Recall that an *unframed* monopole is a solution to $\mathcal{B}(A,\Phi) = 0$ with no boundary value prescribed, modulo the full gauge group $\mathcal{G} = \mathrm{Aut}(P)$, while a *framed* monopole is a solution with prescribed boundary value (A_0,Φ_0) on ∂X, modulo $\mathcal{G}_0 = \{g \in \mathcal{G} : g|\partial X = 1\}$ as discussed above. We denote these moduli spaces by $\mathcal{N}_k(X)$ and $\mathcal{M}_k(X)$, respectively. In the classical case, $\mathcal{N}_k(\mathbb{R}^3)$ is the quotient of $\mathcal{M}_k(\mathbb{R}^3)$ by a free $\mathrm{U}(1)$ action.

In [Kot15c] the complex (1.2) is completed to a Hilbert complex with respect to certain weighted L^2 spaces, and by varying the weight one may consider either framed or unframed deformations.

Theorem 1.3 ([Kot15c])**.** *Let X be an asymptotically conic 3-manifold. There exist natural domains for D_0 and D_1 in a range of weighted L^2 spaces such that (1.2) is an elliptic Hilbert complex. For appropriate weights, the virtual dimensions of $\mathcal{N}_k(X)$ and $\mathcal{M}_k(X)$ are given by the Fredholm index of the Hodge operator $D_0^* + D_1$ on these domains, giving*

$$\begin{aligned}\operatorname{vdim}\mathcal{N}_k(X) &= 4k + \tfrac{1}{2}b^1(\partial X) - b^0(\partial X),\\ \operatorname{vdim}\mathcal{M}_k(X) &= 4k - \tfrac{1}{2}b^1(\partial X).\end{aligned}$$

This is in agreement with the virtual dimension of $\mathcal{N}_k(X)$ computed in [Bra89] for asymptotically hyperbolic X, a different type of geometry, and it represents the first proof of the classical results $\dim\mathcal{N}_k(\mathbb{R}^3) = 4k-1$ and $\dim\mathcal{M}_k(\mathbb{R}^3) = 4k$, to employ a Callias-type index theorem. The difference $\operatorname{vdim}\mathcal{M}_k(X) - \operatorname{vdim}\mathcal{N}_k(X) = b^0(\partial X) - b^1(\partial X)$ is accounted for by considering the moduli space of monopole boundary data and the subgroup of $\mathcal{G}(X)/\mathcal{G}_0(X)$ which fixes such data.

Subsequently, in [Oli14], Oliveira proved the existence of an open set in $\mathcal{N}_k(X)$ of the predicted dimension in the case that X has trivial degree 2 cohomology. However, the *deformation problem*, which is essentially the question of whether $\mathcal{N}_k(X)$ or $\mathcal{M}_k(X)$ is smooth at all points, remains open; see below.

The starting point for the proof of Theorem 1.3 is an index theorem proved in [Kot15a] for operators of the form $D+\Psi$ where D is a self-adjoint Dirac operator and Ψ is a skew-adjoint potential which may have nontrivial nullspace at infinity. This improved on the classical theorem [Cal78] of Callias and its subsequent generalizations in [Ang90, Ang93, Råd94, Bun95]. I generalized the Callias theorem in another direction in [Kot11], replacing D by a pseudodifferential operator and proving a families version of the index theorem in the spirit of [AS68b] using topological K-theory.

Future work. The deformation problem for $\mathcal{M}_k(X)$ and $\mathcal{N}_k(X)$ is to prove the triviality of the *obstruction space*, which is the cokernel of D_1 in (1.2). A straightforward argument shows that the obstruction space vanishes if X has non-negative Ricci curvature; however, among asymptotically conic 3-manifolds, only $\mathbb{R}^3$ has non-negative Ricci curvature, so this is not of much use.

Conjecture 1.4. *If $b^2(X) = 0$, then the obstruction space of (1.2) vanishes for a generic set of asymptotically conic metrics.*

Furthermore, it should be possible to extend most of the analytical methods I and my collaborators have developed above to the setting of monopoles on other spaces and/or with other structure groups besides SU(2). In particular, Murray and Singer have conjectured in [MS03a] that, for a general compact connected Lie group G, in the moduli space of appropriately framed G-monopoles over $\mathbb{R}^3$ should be hyperKähler. They indicate that the spaces have dimensions which are multiples of 4, but are unable to directly analyze the L^2 metric. A natural approach would be to produce a Fredholm extension for the linearization as is done in Theorem 1.3, to which the index theorem in [Kot15a] should apply directly.

Cherkis and Kapustin [CK99, CK01, CK02, CK03] have similar conjectures concerning *periodic monopoles*; i.e., monopoles on $\mathbb{R}^2 \times \mathbb{S}$. They consider charge k monopoles with n singular points, (with structure group U(2) or SO(3)), denoting the moduli space by $\mathcal{M}_{n,k}(\mathbb{R}^2 \times \mathbb{S})$. By way of physical arguments, they compute the metric asymptotics on these spaces and argue that they are complete and hyperKähler; in particular for the case $k = 2$ these give new examples of so-called *gravitational instantons*. While there has been progress on this result in the Ph.D. thesis of Foscolo [Fos13], in which he proved existence and smoothness of the spaces for large values of the mass parameter, the completeness of the metric and rigorous computation of the dimension remains open. The index theorem in [Kot15a] is not directly applicable in this situation since $\mathbb{R}^2 \times \mathbb{S}$ is not asymptotically conic; however, it should be possible to generalize the result to spaces with *fibered boundary* structure, of which $\mathbb{R}^2 \times \mathbb{S}$ is a typical example. Applying such an index result to the monopole deformation complex would give a rigorous computation of the dimension.

2 Loop spaces

To each smooth manifold M, we may associate the free loop space, $\mathcal{L}M = C^\infty(\mathbb{S}^1; M)$, considered as an infinite dimensional Fréchet manifold. The topology of M is reflected in $\mathcal{L}M$ in particular through the *transgression homomorphism* $H^k(M;\mathbb{Z}) \longrightarrow H^{k-1}(\mathcal{L}M;\mathbb{Z})$ in cohomology, obtained by pulling back via evaluation to the product $\mathbb{S}^1 \times \mathcal{L}M$ and pushing forward via the projection to $\mathcal{L}M$.

While $\mathcal{L}M$ itself is is not regarded as a moduli space, there are group actions and equivalence relations on loops which are natural to consider. For instance, the abelian group $\mathrm{U}(1)$, and more generally, the diffeomorphism group $\mathrm{Dff}(\mathbb{S}^1)$, acts by precomposing on the domain space $\mathbb{S}^1$; many of the structures of interest on $\mathcal{L}M$ are equivariant with respect to this action. Rather than work on the extremely singular and still infinite dimensional quotient space $\mathcal{L}M/\mathrm{Dff}(\mathbb{S}^1)$, it is convenient to work equivariantly on $\mathcal{L}M$ itself.

Dirac operators on loop space. In [Wit88] Witten gave a formula for the $\mathrm{U}(1)$-equivariant index of the (formally defined) 'Dirac-Ramond operator', $D_{\mathcal{L}M}$, on the loop space of a compact spin manifold M. The result was obtained by formally invoking the Atiyah-Segal formula [AS68a] to localize to the fixed points of $\mathrm{U}(1)$ acting by loop rotation—which are the constant loops giving an embedding $M \subset \mathcal{L}M$. Witten observed that for manifolds with $p_1(M) = 0$, the index, as a formal power series in characters of $\mathrm{U}(1)$, is valued in modular forms and computes a topological genus of M. Now known as the 'Witten genus,' this cobordism invariant was later shown to be associated to an extraordinary cohomology theory called 'topological modular forms' (or 'tmf'), as the $\widehat{A}$-genus is associated to topological K-theory [AHS01, Lur09]. More generally (see [Wit87]), twisted versions of a Dirac operator have formal indices which compute elliptic genera [Lan88] of M, which are in turn related to the elliptic cohomology theories defined in [LRS95], of which tmf is a universal version.

Taubes in [Tau89] obtained a rigorous formula for Witten's index in terms of twisted Dirac operators on M, by analyzing the formal neighborhood of $M \subset \mathcal{L}M$; however, the analytical tools necessary to make sense of Witten's index theorem on the loop space itself have not yet been developed. My project with Richard Melrose aims to remedy this, by constructing Dirac operators unambiguously as differential operators on $\mathcal{L}M$.

Lithe regularity. To allow for subsequent analysis of the operators, we take full advantage of the fact that $\mathcal{L}M$ is modeled on a space of smooth functions, and work with geometric objects (functions, bundles, sections of bundles, etc.) satisfying a strong regularity condition which we call *litheness* (see [KM13]). To wit, $\mathcal{L}M$ is modeled on the space $C^\infty(\mathbb{S}^1;\mathbb{R}^n)$, $n = \dim(M)$, and the derivative df_p of a lithe function $f \in C^\infty(\mathcal{L}M;\mathbb{R})$—which is a priori a distribution in $C^{-\infty}(\mathbb{S}^1;\mathbb{R}^n) = (C^\infty(\mathbb{S}^1;\mathbb{R}^n))^* \cong T^*_p\mathcal{L}M$—is required to be smooth. There are similar conditions on higher derivatives as well as appropriate definitions for bundles and sections, and the transition diffeomorphisms between smooth charts on $\mathcal{L}M$ is shown to preserve this structure. An example of a lithe function is the holonomy of a $\mathrm{U}(1)$-bundle with connection on M, considered as a $\mathrm{U}(1)$-valued function on $\mathcal{L}M$. The notion is similar, though not identical to the definition of 'super-smooth' objects appearing in [Bry93]. Another approach to the regularity problem inherent in pairing the tangent and cotangent bundles on $\mathcal{L}M$ was developed by Stacey in [Sta08b, Sta08a]; however this involves initially passing to a structure group based on polynomial loops, which is not invariant under $\mathrm{Dff}^+(\mathbb{S}^1)$.

If $P_{\mathrm{Spin}} \longrightarrow M$ is the principal structure bundle of a spin manifold M, the bundle $\mathcal{L}P_{\mathrm{Spin}}$ forms a principal bundle over $\mathcal{L}M$ with structure group $\mathcal{L}\mathrm{Spin} = C^\infty(\mathbb{S}^1; \mathrm{Spin}(2n))$. The ingredients for a Dirac operator are the following:

I) the existence of a *loop-spin structure*, i.e., a lift $\widehat{\mathcal{L}P} \longrightarrow \mathcal{L}P_{\mathrm{Spin}}$ over $\mathcal{L}M$ with structure group $\widehat{\mathcal{L}\mathrm{Spin}}$, the universal central extension of $\mathcal{L}\mathrm{Spin}$ by $\mathrm{U}(1)$.

II) a positive energy representation E of $\widehat{\mathcal{L}\mathrm{Spin}}$, from which is formed the associated *spinor* bundle $\mathbb{E} \longrightarrow \mathcal{L}M$,

III) a 'Clifford action' $c\ell : T\mathcal{L}M \longrightarrow \mathrm{End}(\mathbb{E})$, and

IV) a smoothly differentiable connection ∇ on $\mathbb{E}$ compatible with the Clifford action.

The theory of central extensions of $\mathcal{L}G$ for Lie groups G is well-known, [PS88]; however the standard construction of $\widehat{\mathcal{L}G}$ is as a quotient of Hilbert groups, which we avoid in [KM13] by constructing $\widehat{\mathcal{L}G}$ inside an algebra of Toeplitz-type pseudodifferential operators.

The existence problem I) for loop-spin structures is topological. Indeed, $\mathcal{L}P_{\mathrm{Spin}}$ and the central extension $\widehat{\mathcal{L}\mathrm{Spin}}$ give rise to a lifting bundle gerbe [Mur96, MS03b] on $\mathcal{L}M$ with an associated cohomology class $\sigma \in H^3(\mathcal{L}M; \mathbb{Z})$, the nonvanishing of which obstructs the existence of a lift. From a result of McLaughlin [McL92], if $\dim(M) \geq 5$ then σ is the transgression of the class $\frac{1}{2}p_1(M) \in H^4(M; \mathbb{Z})$. Thus M is called a *string manifold* provided it is oriented, spin, and satisfies $\frac{1}{2}p_1(M) = 0$.

Fusion. The identification of such lifts is a more difficult question, and has to do with the problem of characterizing those geometric objects, such as $\mathrm{U}(1)$ bundles on $\mathcal{L}P_{\mathrm{Spin}}$, on loop spaces which are related via transgression to geometric objects, such as gerbes on P_{Spin}, on the base manifold. In [Wal12b, Wal10, Wal12c, Wal12a, Wal14], Waldorf identified 'fusion' as a key property. Originally introduced by Stolz and Teichner in [ST] with respect to functions, fusion is a multiplicativity property involving the identification $\mathcal{L}M \subset \mathcal{I}^{[2]}M$ of loops with pairs in the path space $\mathcal{I}M = C^\infty([0,1]; M)$ having the same endpoints, and the pullbacks of this space to $\mathcal{I}^{[3]}M$. Here $\mathcal{I}M \longrightarrow M^2$ is a fibration with respect to the evaluation map at the endpoints, and $\mathcal{I}^{[k]}M$ denotes the k-fold fiber product of this fibration with itself. For example, a $\mathrm{U}(1)$-bundle $L \longrightarrow \mathcal{L}M$ is fusion provided its extension by continuity to $\mathcal{I}^{[2]}M \supset \mathcal{L}M$ is equipped with an isomorphism

$$\pi_{12}^*L \otimes \pi_{23}^*L \cong \pi_{13}^*L \text{ on } \mathcal{I}^{[3]}M, \tag{2.1}$$

which is associative over $\mathcal{I}^{[4]}M$, where $\pi_{ij} : \mathcal{I}^{[3]}M \longrightarrow \mathcal{I}^{[2]}M$ denote the fiber projections. A fusion function satisfies a direct multiplication identity along the lines of (2.1).

In [KM13] we define a *fusive* structure to consist of fusion multiplicativity, $\mathrm{Dff}^+(\mathbb{S}^1)$-equivariance and lithe regularity, and we define $H^1_{\mathrm{fus}}(\mathcal{L}M; \mathbb{Z})$ and $H^2_{\mathrm{fus}}(\mathcal{L}M; \mathbb{Z})$ to be equivalence classes of fusive $\mathrm{U}(1)$-functions and fusive $\mathrm{U}(1)$-bundles, respectively.

Theorem 2.1 ([KM13])**.**

(a) *For $k = 1, 2$, there are 'enhanced transgression' isomorphisms $H^{k+1}(M; \mathbb{Z}) \xrightarrow{\cong} H^k_{\mathrm{fus}}(\mathcal{L}M; \mathbb{Z})$, which when composed with the natural map $H^k_{\mathrm{fus}}(\mathcal{L}M; \mathbb{Z}) \longrightarrow H^k(\mathcal{L}M; \mathbb{Z})$ forgetting the fusive structure, coincide with the transgression map in cohomology.*

(b) *If $\frac{1}{2}p_1(M) = 0 \in H^4(M; \mathbb{Z})$, then fusive loop-spin structures $\widehat{\mathcal{L}P} \longrightarrow \mathcal{L}P_{\mathrm{Spin}}$ are classified by $H^3(M; \mathbb{Z})$. (c.f. [Wal14]).*

In a related but slightly different direction, one can drop the regularity and $\mathrm{Diff}^+(\mathbb{S}^1)$ requirements and try to determine a structure on $\mathcal{L}M$ sufficient to characterize the image of transgression from M. In [KM15b] we do this at the level of Čech cohomology on the continuous loop space $\mathcal{L}_{\mathcal{C}}M = C^0(\mathbb{S}^1; M)$, defining *loop-fusion* cohomology groups $\check{H}^k_{\mathrm{lf}}(\mathcal{L}_{\mathcal{C}}M; A)$, for any degree $k \geq 1$ and abelian group A, in terms of Čech cochains on a good open cover of $\mathcal{L}_{\mathcal{C}}M$ which are multiplicative with respect to fusion as well as a second, 'figure-of-eight' product on loops:

Theorem 2.2 ([KM15b]). *There is an enhanced transgression isomorphism*

$$T_{\mathrm{lf}} : \check{H}^k(M; A) \xrightarrow{\cong} \check{H}^{k-1}_{\mathrm{lf}}(\mathcal{L}_{\mathcal{C}}M; A) \tag{2.2}$$

through which the standard transgression map $T : \check{H}^k(M; A) \longrightarrow \check{H}^{k-1}(\mathcal{L}_{\mathcal{C}}M; A)$ factors.

Future work. One problem is to refine Theorem 2.2 to the smooth loop space and combine it with a de Rham version of the enhanced transgression, which would lead to a version of Theorem 2.2 for differential cohomology:

Problem 2.3. *Define an appropriate 'loop-fusion' version of differential cohomology $\widehat{H}^k(\mathcal{L}M; \mathbb{Z})$ and extend (2.2) to an isomorphism $\widehat{H}^k(M; \mathbb{Z}) \xrightarrow{\cong} \widehat{H}^{k-1}_{\mathrm{lf}}(\mathcal{L}M; \mathbb{Z})$. Give a geometric interpretation of this transgression in terms of higher gerbes with connection.*

Ingredients II)-IV) for Dirac operators on $\mathcal{L}M$ are the subject of ongoing work. The positive energy representation theory of central extensions $\widehat{\mathcal{L}G}$ is well-developed (see for instance [PS88] and [TL99]); however, appropriately regular (i.e., lithe) versions of these representations have yet to be defined, along with the Clifford action of $T\mathcal{L}M$.

To summarize, in the first phase of our project we will prove

Conjecture 2.4. *Provided M is string, $\mathcal{L}P_{\mathrm{Spin}}$ extends to a fusive principal bundle $\widehat{\mathcal{L}P} \longrightarrow \mathcal{L}M$ with structure group $\widehat{\mathcal{L}\mathrm{Spin}}$ with a lithe connection ∇ which extends the Levi-Civita connection. For an appropriate positive energy representation E of $\widehat{\mathcal{L}\mathrm{Spin}}$, there is an associated 'spinor' bundle $\mathbb{E} = \widehat{\mathcal{L}P} \times_{\widehat{\mathcal{L}\mathrm{Spin}}} E$ which is fusive, and admits a fusive Clifford action of $T\mathcal{L}M$. The resulting Dirac operator $D_{\mathcal{L}M} = c\ell \circ \nabla$ is well-defined as a differential operator on lithe sections of $\mathbb{E}$.*

The next problem is to understand the nullspace of $D_{\mathcal{L}M}$, using the torus case $M = \mathbb{T}^n$ as a guide, where representation theory can be brought to bear. In [FHT13], Freed, Hopkins and Teleman define G-equivariantly Fredholm Dirac operators on $\mathcal{L}G$ for a group G and use representation theory and families of such operators to construct an isomorphism to the twisted equivariant K-theory of G; however, these operators do not include the Dirac-Ramond operator or its variants.

Problem 2.5. *Determine the regularity properties of $\mathrm{Null}(D_{\mathcal{L}M})$ for a Dirac operator $D_{\mathcal{L}M}$, and show that it is finite dimensional at each energy level, i.e., character of the $\mathrm{U}(1)$ action. Deduce that it is $\mathrm{U}(1)$ (or more generally $\mathrm{Dff}^+(\mathbb{S}^1)$) equivariantly Fredholm as an operator on lithe sections.*

Following this, one would like to understand the topological side of Witten's index formulas, which should amount to statements in elliptic cohomology/topological modular forms (see [Lur09] for a survey). An adequate 'geometric' characterization of $\mathrm{tmf}^*(M)$ for a manifold M has yet to be carried out, though it is the subject of speculation and ongoing work by several authors [Seg88, ST04, DH11]. The structures on $\mathcal{L}M$ mentioned above, in particular fusion and the $\mathrm{Dff}^+(\mathbb{S}^1)$ action, suggest various intriguing possibilities; indeed, it has been suggested that some kind of $\mathrm{Dff}^+(\mathbb{S}^1)$-equivariant K-theory of $\mathcal{L}M$ could potentially give such a geometric characterization [Bry90, DH11]. The following would be a major result:

Problem 2.6.

(a) *Determine the relationship between the fusion and* $\mathrm{Dff}^+(\mathbb{S}^1)$ *structures on* $\mathcal{L}M$ *on the one hand and* $\mathrm{tmf}^*(M)$ *on the other.*

(b) *Construct a topological index map for the Dirac-Ramond operator in this setting and prove Witten's index theorem.*

3 Manifolds with generalized corners and blow-up

Manifolds with corners have proved to be a convenient setting for many analysis problems on singular and/or non-compact spaces. The latter may often be resolved and/or compactified to manifolds with corners, after which the rich analytical theory developed by Melrose and his students may be applied. The category of manifolds with corners and so-called 'b-maps' has many parallels with the category of manifolds without boundary. However, there are some subtleties, particularly with respect to fiber products, and it is becoming apparent that a slightly larger category of manifolds with 'generalized corners' may be useful.

Fiber products in manifolds with ordinary corners. Motivated by a result [Joy12] of Joyce on fiber products in a category of manifolds with corners under a more restricted class of maps, Richard Melrose and I posed the following question: Given b-maps $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ between manifolds with corners, under what conditions is the *fiber product*

$$X \times_Z Y = \{(x, y) : f(x) = g(y)\} \subset X \times Y \tag{3.1}$$

a smooth manifold with corners?

To answer this question, we borrowed ideas from toric algebraic geometry. To each boundary face $F \subset X$ of a manifold with corners, there is a naturally associated *monoid* $Q_F \cong \mathbb{N}_0^{\mathrm{codim}(F)}$, and whenever $G \subset F$, there is an injective homomorphism $Q_F \longrightarrow Q_G$. Together these form what we call a *monoidal complex* $\mathcal{P}_X$, and every (interior) b-map $f : X \longrightarrow Y$ induces a morphism $f_\natural : \mathcal{P}_X \longrightarrow \mathcal{P}_Y$ of complexes.

Theorem 3.1 ([KM15a])**.** *If f and g are b-transverse interior b-maps and if every monoid in the complex $\mathcal{P}_X \times_{\mathcal{P}_Z} \mathcal{P}_Y$ is freely generated, then (3.1) is a smooth manifold with corners with the universal property of the fiber product: any manifold W with interior b-maps to X and Y forming a commutative square with Z factors uniquely through $X \times_Z Y$.*

The b-transversality condition is the analogue in the category of manifolds with corners of the classical transversality condition in the boundaryless case. Unfortunately, among b-transverse maps the second condition is rarely satisfied, and then $X \times_Z Y$ is too singular to be a manifold with corners.

However, it is a type of space with similar properties: it is stratified by boundary faces of the same type, and likewise supports a monoidal complex $\mathcal{P}_{X \times_Z Y}$ in which the monoids need not be freely generated. In this case, we showed also in [KM15a] that for any freely generated *refinement* $\mathcal{R} \longrightarrow \mathcal{P}_{X \times_Z Y}$ (essentially a consistent subdivision of each of the monoids in the target) there exists a unique resolution, or *blow-up* of $X \times_Z Y$ to a manifold with corners whose complex is isomorphic to $\mathcal{R}$ (c.f. Theorem 3.2 below).

Generalized corners. In response to our result, Joyce developed the category of *manifolds with generalized corners* in [Joy15], and showed that it is closed under b-transverse fiber products. A manifold with generalized corners is a space locally modeled on the spaces $\mathrm{Hom}(P; [0, \infty))$, of monoid homomorphisms from a *toric monoid* P to the (multiplicative) monoid $[0, \infty)$. In the case that $P \cong \mathbb{N}_0^k$ is freely generated, $\mathrm{Hom}(P, [0,\infty)) \cong [0,\infty)^k$, and the definition of a manifold with corners is recovered. Manifolds with generalized corners are quite similar to their classical counterparts, though they allow for more complicated behavior at the corners. For instance, while a classical corner of codimension k must be locally modeled by the intersection of k boundary hypersurfaces, a generalized corner may be modeled by, for example, the space $\{(x_1, x_2, x_3, x_4) : x_1x_2 = x_3x_4\} \subset [0,\infty)^4$, in which 4 boundary hypersurfaces meet at a point with codimension 3.

There is also an algebro-geometric theory of Gillam and Molcho developed in [GM15] (also written in response to [KM15a]), in which in which manifolds with corners arise as a natural subcategory of the category of 'positive log differentiable spaces'. In this formulation, the 'b-' objects associated to manifolds with corners as defined by Melrose are the natural ones corresponding to a 'logarithmic structure' on such a space, in the sense of [KKMSD73, Kat94].

One of the principal developments of [KM15a] was the generalization of the real blow-up of boundary faces associated with a refinement of monoidal complexes, and in [Kot15b] I extended this to the setting of manifolds with generalized corners.

Theorem 3.2 ([Kot15b])**.** *There is a covariant functor $X \longmapsto \mathcal{P}_X$, $(f : X \to Y) \longmapsto (f_\natural : \mathcal{P}_X \to \mathcal{P}_Y)$ from the category of manifolds with generalized corners to complexes of toric monoids.*

(a) *To each (not necessarily freely generated) refinement $\psi : \mathcal{R} \longrightarrow \mathcal{P}_X$, there is a unique* blow-up*, i.e., a manifold with generalized corners $[X;\mathcal{R}]$ and a* blow-down *map $\beta : [X;\mathcal{R}] \longrightarrow X$, which is a diffeomorphism on interiors and such that $\mathcal{P}_{[X;\mathcal{R}]} \cong \mathcal{R}$ and $\beta_\natural \cong \psi$.*

(b) *The blow-up satisfies the following universal property: If the morphism $f_\natural : \mathcal{P}_Y \longrightarrow \mathcal{P}_X$ associated to a map $f : Y \longrightarrow X$ factors through $\psi : \mathcal{R} \longrightarrow \mathcal{P}_X$, then f factors through a unique map $\widetilde{f} : Y \longrightarrow [X;\mathcal{R}]$.*

(c) *Blow-up is stable under pullback; i.e., if $f : Y \longrightarrow X$ is any interior b-map, then $Y \times_X [X;\mathcal{R}] \cong [Y;\mathcal{R}']$, where $\mathcal{R}' = \mathcal{P}_Y \times_{\mathcal{P}_X} \mathcal{R}$ is a refinement of $\mathcal{P}_Y$.*

Future work. The category of manifolds with generalized corners appears to be rather well-behaved geometrically, and I expect that it has an important role to play in the theory of moduli spaces, in that various moduli spaces (in particular, moduli spaces of monopoles) should have natural compactifications and/or resolutions as manifolds with generalized corners.

However, the utility of generalized corners for problems in analysis will ultimately depend on the extent to which the analytical tools of Melrose can be extended from ordinary to generalized corners. One particularly important piece is the following. Recall that a *polyhomogeneous* function on a manifold with corners is a function, u, which is smooth on the interior and has a complete asymptotic expansion at each boundary hypersurface of the form

$$u \sim \sum_{(z,k)\in E} \rho^z (\log \rho)^k u_{z,k},$$

where $u_{z,k}$ are again polyhomogeneous coefficients on the hypersurface itself, ρ is a boundary defining function, and E is a suitable discrete subset of $\mathbb{C} \times \mathbb{N}_0$. This class of functions is stable under pullback by interior b-maps, and even more importantly, is also stable under pushforward (i.e., fiber integration) with respect to so-called 'b-fibrations' (which are the natural generalization of fiber bundles in the category of manifolds with corners) [Mel92].

Problem 3.3. *Is there a well-defined class of polyhomogeneous functions on a manifold with generalized corners which is stable under pullback by b-maps and pushforward by b-fibrations?*

Such a pullback and pushforward theorem would go a long way toward the development of pseudodifferential operators on manifolds with generalized corners.

One natural question raised by Theorem 3.2 has been posed to me by Dominic Joyce. Namely, the blow-down map β has the property that its b-differential

$$\beta_* : {}^bT_p[X;\mathcal{R}] \longrightarrow {}^bT_{\beta(p)}X$$

is an isomorphism for all p; such a map might be called *b-etále*. The following conjecture holds for manifolds with ordinary corners, as was shown in [KM15a].

Conjecture 3.4. *Every b-etále map between manifolds with generalized corners is locally a blow-down map.*

Finally, note that the blow-ups appearing Theorem 3.2 generalize radial blow-up of boundary faces. However, the utility of radial blow-up in classical setting the is not limited to blow-up of boundary faces, so it would be of considerable interest to extend the theory above, replacing the stratification of X by boundary faces by a more general type of stratification. The finite collection of monoids in $\mathcal{P}_X$ would have to be replaced by something more complicated. One possibility is the notion of a *monoidal space* as defined by Kato [Kat94] in the setting of logarithmic algebraic geometry; see also [GM15] and [ACUW15].

Problem 3.5. *Stratify a manifold X with generalized corners by a collection of submanifolds which is closed under transverse or clean intersection. What kind of structure, analogous to $\mathcal{P}_X$, can you associate to such a stratification which captures the necessary information to generalize radial blow-up of these submanifolds?*

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