Functional Analysis Homework 4, Fall 2016

Problem 1. Let $T \in \mathcal{B}_0(H)$ be a compact, self-adjoint operator on a seperable Hilbert space H, and suppose T is *positive*:

$$\langle Tx, x\rangle \geq 0, \quad \forall\ x \in H.$$

Show that T has only real, non-negative eigenvalues, which can be arranged in a weakly decreasing sequence

$$\lambda_1 \geq \lambda_2 \geq \cdots$$

either finite or with limit 0. (Note: here we represent eigenvalues with multiplicity, i.e., we repeat λ_j according to the dimension of the associated eigenspace.)

Moreover, show that if $M \subset H$ is a subspace of dimension n, then

$$\min_{x\in M, \|x\|=1} \langle Tx, x\rangle \leq \lambda_n, \quad n = \dim(M).$$

Problem 2. With the same hypotheses on T, show that the decreasing sequence of eigenvalues are given by the *minimax formula*

$$\lambda_j(T) = \max_{M\subset H, \dim(M)=n} \Big(\min_{x\in M, \|x\|=1} \langle Tx, x\rangle \Big).$$

Problem 3. Let $A \in \mathcal{B}(H)$ be an operator on a seperable Hilbert space with the property that for some orthonormal basis $\{e_n\}$,

$$\|A\|_{\mathrm{HS}}^2 := \sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty.$$

(a) Show that $\|A^*\|_{\mathrm{HS}}^2$ is also finite. (Hint: show that $\|A\|_{\mathrm{HS}}^2$ is equivalently given by $\sum_{i,j} a_{ij}^2$, where $a_{ij} = \langle Ae_i, e_j\rangle$.)
(b) Show that if $B \in \mathcal{B}(H)$ is a bounded operator then $\|BA\|_{\mathrm{HS}} \leq \|B\|\,\|A\|_{\mathrm{HS}}$.
(c) Show that $\|A\|_{\mathrm{HS}}$ is independent of the choice of orthonormal basis $\{e_n\}$ used above. In particular, if A is normal, then

$$\|A\|_{\mathrm{HS}}^2 = \sum_{n=1}^{\infty} |\lambda_n|^2$$

where $\{\lambda_n\}$ are the eigenvalues of A.
(d) Show that A is a compact operator. (Hint: show it is the norm limit of a sequence of finite rank operators.)
(e) The set $\mathcal{B}_2(H)$ of operators satisfying $\|A\|_{\mathrm{HS}} < \infty$ are called the *Hilbert-Schmidt operators*. We conclude that $\mathcal{B}_2(H)$ is a 2-sided, $*$-closed ideal in $\mathcal{B}(H)$ which is contained inside the compact operators: i.e. $\mathcal{B}_2(H) \subset \mathcal{B}_0(H)$.

Problem 4. Show that, for any choice of orthonormal basis $\{e_n\}$,

$$\langle A, B\rangle_{\mathrm{HS}} = \sum_{n=1}^{\infty} \langle Ae_n, Be_n\rangle$$

is an inner product on $\mathcal{B}_2(H)$ with respect to which it is a Hilbert space.