A short story of measure theory

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1 Introduction

The Riemann integral is founded on the following idea: divide up the domain of a function $f : [a, b] \longrightarrow \mathbb{R}$ into subintervals, estimate f from above and below on each interval, and approximate the integral of f by the *upper* and *lower sums*—the summation of the widths of the intervals times the upper and lower estimates on each. The limit over partitions of $[a, b]$ of these two approximations, should they exist and agree, is declared to be the integral of f.

Sadly, this definition of the integral lacks some desirable properties. In particular, the space of absolutely integrable functions is not complete—a sequence of functions which is Cauchy in the norm $\|f\|_1 = \int_a^b |f(x)|\, dx$ need not converge to a Riemann-integrable function.

As a remedy to such deficiencies, the Lebesgue integral is founded on a different idea: namely, divide up the *range* of f into subintervals, and approximate the integral by the summation of the lower endpoints times the *volume*, or *measure*, of the interval's preimage under f. To make this idea precise, we require

(1) a notion of measure for appropriate sets,

(2) a class of "measurable" functions which can be so approximated, and

(3) a definition of the integral of a measurable function on a measurable set.

As with most ideas in math, it is possible to develop this in a fairly general setting. In this note, we outline this development in the general setting, with particular mention of the Lebesgue measure on $\mathbb{R}^n$. As this is a "story", not a course in measure theory, you are meant to provide your own proofs (or look them up). Most are straightforward, if tedious. Folland's *Real Analysis* is the treatment we mostly follow here.

2 Measures

It is an unfortunate fact that we often cannot assign a coherent measure to *all* subsets of a given space. We can, however, require some nice conditions of those sets to be 'measured'.

A collection $\mathcal{A} \subseteq \mathcal{P}(X)$ of subsets of X is an **algebra** if it contains $\emptyset$ and is closed under pairwise (hence finite) union and complements:

$$A_1,\ A_2 \in \mathcal{A} \implies A_1 \cup A_2,\ A_1^c \in \mathcal{A}.$$

$\mathcal{A}$ is a **σ-algebra** if in addition it is closed under *countable* unions:

$$\{A_n : n \in \mathbb{N}\} \subset \mathcal{A} \implies \bigcup_n A_n \in \mathcal{A}.$$

It follows that $\mathcal{A}$ is likewise closed under countable intersections.

Often we start with a collection of sets of interest, and take the smallest σ-algebra generated by these. If X is a topological space, the **Borel** σ-algebra, $\mathcal{B}_X$, is the one generated by all open (equivalently closed) sets.

Proposition 2.1. *The Borel σ-algebra on $\mathbb{R}$ is equivalently generated by any of the following collections of subsets:*

$$\{(a,b) : a,b \in \mathbb{R}\} \qquad \{[a,b) : a,b \in \mathbb{R}\} \qquad \{(a,b] : a,b \in \mathbb{R}\}$$
$$\{[a,b] : a,b \in \mathbb{R}\} \qquad \{(a,\infty) : a \in \mathbb{R}\} \qquad \{[a,\infty) : a \in \mathbb{R}\}$$
$$\{(-\infty,a) : a \in \mathbb{R}\} \qquad \{(-\infty,a] : a \in \mathbb{R}\}$$

In measure theory it is often useful to work with the **extended real numbers** $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, a 2-point compactification of $\mathbb{R}$ with the obvious topology (i.e., $(a,\infty]$ and $[-\infty,b)$ are open for all $a,b \in \mathbb{R}$) and total order. Then $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by the collection $\{[a,\infty]\}$, for instance.

Let $\mathcal{A}$ be a σ-algebra on a set X. A **measure** on $(X,\mathcal{A})$ is a function $\mu : \mathcal{A} \longrightarrow [0,\infty]$ satisfying

(M1) $\mu(\emptyset) = 0$, and

(M2) (**Countable additivity**) if $\{A_n : n \in \mathbb{N}\}$ are mutually disjoint then $\mu\Big(\bigcup_{n=1}^\infty A_n\Big) = \sum_{n=1}^\infty \mu(A_n)$.

The properties below follow easily from the two defining ones (M1) and (M2).

Proposition 2.2. *Let μ be a measure on $(X,\mathcal{A})$. Then*

(M3) **(Monotonicity)** $A \subset B \implies \mu(A) \le \mu(B)$,

(M4) **(Countable sub-additivity)** $\mu\big(\bigcup_{n=1}^\infty A_n\big) \le \sum_{n=1}^\infty A_n$,

(M5) **(Continuity from below)** $A_1 \subset A_2 \subset \cdots \implies \mu\big(\bigcup_n A_n\big) = \lim_n \mu(A_n)$,

(M6) **(Continuity from above)** $A_1 \supset A_2 \supset \cdots \implies \mu\big(\bigcap_n A_n\big) = \lim_n \mu(A_n)$.

We defer the existence and construction of useful measures until §6.

3 Measurable functions

Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be spaces with σ-algebras (aka "measurable spaces"). A function $f : X \longrightarrow Y$ is **measurable** if

$$B \in \mathcal{B} \implies f^{-1}(B) \in \mathcal{A}.$$

In particular, a (possibly extended) real-valued function $f : X \longrightarrow \overline{\mathbb{R}} = (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$ is measurable if and only if $f^{-1}([a,\infty]) \in \mathcal{A}$ for all $a \in \mathbb{R}$. The set of measurable $\overline{\mathbb{R}}$-valued functions has particularly nice limit properties:

Proposition 3.1. *Let (f_n) be a sequence of $\overline{\mathbb{R}}$-valued measurable functions on $(X,\mathcal{A})$. Then*

$$g_1(x) = \sup_n f_n(x), \quad g_2(x) = \inf_n f_n(x),$$
$$g_3(x) = \limsup_n f_n(x), \quad \textit{and } g_4(x) = \liminf_n f_n(x)$$

are all measurable. In particular if the sequence converges pointwise then $\lim_n f_n$ is measurable.

A **step function** is a measurable function given by a finite linear combination

$$\phi = \sum a_k \chi_{A_k}, \quad A_k \in \mathcal{A}, \quad a_k \in \mathbb{C},$$

where χ_A denotes the **indicator function**

$$\chi_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

(Note that the a_k are not allowed to be infinite, and that by requiring the A_k to be disjoint, we can arrange for a unique representation of ϕ.) For a step function, the definition of the integral is almost obvious; however we run into issues whenever some of the A_k have infinite measure.

Initially then, we restrict attention to the *positive* measurable functions:

$$\mathcal{L}^+ = \mathcal{L}^+(X) = \{f : X \longrightarrow [0, \infty] \text{ measurable}\}.$$

Proposition 3.2. *$f \in \mathcal{L}^+$ if and only if there is an increasing sequence of positive step functions (ϕ_n) such that $\phi_n \to f$ pointwise.*

4 The integral

For a positive step function $\phi = \sum a_k \chi_{A_k}$, $a_k \in [0, \infty)$, the integral is defined by

$$\int \phi \, d\mu = \sum a_k \mu(A_k), \tag{1}$$

with the convention that $0 \cdot \infty = 0$. Note that $\int \phi \, d\mu$ may have the value ∞.

Proposition 4.1. *The integral (on step functions) has the following properties:*

(a) $\int (\phi + \psi) \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu$.

(b) $\int c\phi \, d\mu = c \int \phi \, d\mu$, $c \in [0, \infty)$.

(c) *If $\phi \leq \psi$, then $\int \phi \, d\mu \leq \int \psi \, d\mu$.*

(d) *$A \longmapsto \int_A \phi \, d\mu = \sum a_k \mu(A \cap A_k)$ is a measure on $\mathcal{A}$.*

For a positive measurable function $f \in \mathcal{L}^+$, the integral is defined by estimating from below by step functions:

$$\int f \, d\mu := \sup \left\{ \int \phi \, d\mu : 0 \leq \phi \leq f, \ \phi \text{ step} \right\}$$

This extends (1) when f is a step function, since the supremum is then achieved by $\phi = f$.

Theorem 4.2 (Monotone Convergence Theorem)**.** *Let (f_n) be a sequence in $\mathcal{L}^+$ such that $f_n \leq f_{n+1}$ for all n and $f_n \to f \in \mathcal{L}^+$. Then*

$$\int f \, d\mu = \lim_n \int f_n \, d\mu.$$

Instead of taking the supremum over all step functions bounded by $f \in \mathcal{L}^+$, we can thus represent each f by a pointwise increasing limit of step functions by Proposition 3.2 and exchange limits and integral signs by Theorem 4.2.

Corollary 4.3. *Proposition 4.1 extends to the integral on $\mathcal{L}^+$; in fact the latter is countably additive: $\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$.*

Note that, without the monotone increasing hypothesis, Theorem 4.2 may fail. For instance, $f_n = \chi_{[n,n+1]}$ and $g_n = n\chi_{[0,1/n]}$ are two sequences of step functions on $\mathbb{R}$ converging pointwise to 0, but for which $\int f_n \, dx = \int g_n \, dx = 1$ for all n. A general inequality holds however:

Corollary 4.4 (Fatou's Lemma)**.** *Let (f_n) be any sequence in $\mathcal{L}^+$. Then*

$$\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n \, d\mu$$

We are tempted to suppose that $0 \leq f$, $\int f \, d\mu = 0$ implies $f = 0$, but this is generally false, as can be seen already for step functions. Indeed, if $\phi = a\chi_A$ where the A has measure zero ($\mu(A) = 0$), then $\int \phi \, d\mu = 0$ even if $a \neq 0$. We say that a property that holds off of a set of measure zero holds **almost everywhere**, or **a.e.**, for short[1].

Proposition 4.5. *If $f \in \mathcal{L}^+$ and $\int f \, d\mu = 0$, then $f = 0$ almost everywhere.*

Evidently we are free to alter measurable functions on a set of measure zero without altering their integrals. It follows that Theorem 4.2 holds under the relaxed condition that $f_n \nearrow f$ pointwise a.e. (hereafter we just say "$f_n \nearrow f$ a.e."), rather than pointwise everywhere.

5 Integrating real and complex functions

If f is a $\mathbb{R}$-valued measurable function, then $f = f_+ - f_-$ where $f_+ = \max(f, 0)$ and $f_- = -\min(f, 0)$ are measurable (c.f. Prop. 3.1) and positive. Note that $|f| = f_+ + f_-$ is also measurable and positive. We say f is **integrable** if

$$\int |f| \, d\mu < \infty,$$

which implies that both $\int f_+ \, d\mu$ and $\int f_- \, d\mu$ are finite, and we define

$$\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu.$$

We denote the set of real valued integrable functions by $\mathcal{L}(X;\mathbb{R})$.

Proposition 5.1. *$\mathcal{L}(X,\mathbb{R})$ is a vector space, $\int \cdot \, d\mu : \mathcal{L}(X;\mathbb{R}) \longrightarrow \mathbb{R}$ is a linear functional, and $\left|\int f \, d\mu\right| \leq \int |f| \, d\mu$.*

Likewise, we say a complex valued function g is integrable if $\int |g| \, d\mu < \infty$, which holds if and only if $\operatorname{Re} g$ and $\operatorname{Im} g$ are integrable real functions, and we define

$$\int g \, d\mu = \int \operatorname{Re} g \, d\mu + i \int \operatorname{Im} g \, d\mu.$$

Denote the set of complex valued integrable functions by $\mathcal{L}(X;\mathbb{C})$. Proposition 5.1 extends to $\mathcal{L}(X;\mathbb{C})$.

The workhorse limit theorem in Lebesgue integration theory is the following.

[1] Given a measure space $(X, \mathcal{A}, \mu)$, it is technically useful to suppose that $\mathcal{A}$ contains all subsets of sets of μ measure 0, (which should have measure 0 by monotonicity), and this can always be arranged by enlarging $\mathcal{A}$. Such a μ is said to be **complete**.

Theorem 5.2 (Lebesgue Dominated Convergence Theorem)**.** *Let (f_n) be a sequence in $\mathcal{L}(X;\mathbb{C})$ such that $f_n \to f$ pointwise a.e., and suppose there exists a real valued $g \geq 0$ with $\int g\, d\mu < \infty$ and $|f_n| \leq g$ for all n. Then f is integrable and*

$$\int f\, d\mu = \lim_n \int f_n\, d\mu.$$

6 Construction of measures

How do we come up with useful measures in practice? One way is to start with a putative measure defined on some collection of sets, not necessarily a σ-algebra, and try to extend it.

For example, the starting point for Lebesgue measure in $\mathbb{R}^n$ is the standard volume of a "rectangle" $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$, which is $\lambda(A) = \prod_{i=1}^n (b_i - a_i)$. We can extend λ additively to the set $\mathcal{A}$ of countable disjoint unions of such rectangles. Then $\lambda(\emptyset) = 0$ and it is countably additive, but $\mathcal{A}$ is not a σ-algebra as it is not closed under complements, so we wish to extend λ to a measure on some σ-algebra which contains $\mathcal{A}$. (Note that any such σ-algebra will contain the σ-algebra generated by $\mathcal{A}$, which is the Borel algebra $\mathcal{B}_{\mathbb{R}^n}$).

Suppose more generally that $\lambda : \mathcal{A} \longrightarrow [0, \infty]$ satisfies the conditions of a measure for some collection $\mathcal{A}$ of subsets of X, not necessarily a σ-algebra, but closed under disjoint countable unions. For an arbitrary subset $E \subset X$, we define

$$\lambda^*(E) = \inf \{\lambda(A) : E \subset A,\ A \in \mathcal{A}\} \tag{2}$$

Then $\lambda^* : \mathcal{P}(X) \longrightarrow [0, \infty]$ may not be a measure, but it satisfies the weaker properties of a so-called **outer measure**:

(M1) $\lambda^*(\emptyset) = 0$

(M3) $E \subset F \implies \lambda^*(E) \leq \lambda^*(F)$

(M4) $\lambda^*(\bigcup_n E_n) \leq \sum_n \lambda^*(E_n)$

An arbitrary subset $F \subset X$ is said to be **λ^*-measurable** if

$$\lambda^*(E) = \lambda^*(E \cap F) + \lambda^*(E \cap F^c) \quad \text{for every set } E \subset X. \tag{3}$$

Note that, in the particular case that $E \in \mathcal{A}$ is a basic set containing F, $\lambda^*(E \cap F) = \lambda^*(F)$ is the outer measure of F, while $\lambda^*(E) - \lambda^*(E \cap F^c)$ is a kind of "inner measure" of F—the measure of the best approximation of F from the inside. Then (3) says that these agree if F is measurable. For technical reasons it is necessary to demand (3) hold for all sets E.

Theorem 6.1 (Carathéodory's Theorem)**.** *Suppose $\lambda^* : \mathcal{A} \longrightarrow [0, \infty]$ is an outer measure (i.e., satisfies (M1), (M3) and (M4) above). Then the collection $\mathcal{M}$ of λ^*-measurable sets is a σ-algebra and λ^* is a complete measure on $\mathcal{M}$.*

Applying Carathéodory's Theorem to the outer measure defined by (2), where λ is the standard volume on the set $\mathcal{A}$ of countable disjoint unions of rectangles in $\mathbb{R}^n$, leads to **Lebesgue measure** $(\mathbb{R}^n, \mathcal{M}, \lambda)$ on $\mathbb{R}^n$. There is no particularly nice characterization of the Lebesgue measurable sets $\mathcal{M}$; it is strictly larger than the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^n}$ on the one hand, yet it is not all of $\mathcal{P}(\mathbb{R}^n)$ on the other hand. Indeed, results such as the Banach-Tarski Paradox imply the existence of Lebesgue unmeasurable sets.

The key property of $(\mathbb{R}^n, \mathcal{M}, \lambda)$ is its behavior with respect to translations, dilations, and rotations.

Proposition 6.2. *If $E \in \mathcal{M}$ and $s \in \mathbb{R}^n$, then $E + s \in \mathcal{M}$ and $\lambda(E + s) = \lambda(E)$. Likewise $aE \in \mathcal{M}$ for $a \in \mathbb{R}$ and $\lambda(aE) = |a|\, \lambda(E)$. Finally, if $T \in \mathrm{O}(n)$ is an orthogonal transformation ($n \times n$ matrix with $T^*T = I$) , then $T(E) \in \mathcal{M}$ and $\lambda\big(T(E)\big) = \lambda(E)$.*

7 L^p spaces

We would like to equip $\mathcal{L}(X;\mathbb{C})$ with a norm given by integration; however, from Proposition 4.5 $\int |f|\, d\mu = 0$ only implies that $|f| = 0$, and hence $f = 0$ holds almost everywhere—off of a set of measure zero. For this reason set

$$L^1(X;\mathbb{C}) = \mathcal{L}(X;\mathbb{C})/\mathcal{Z}, \quad \mathcal{Z} = \{f \in \mathcal{L}(X;\mathbb{C}) : f = 0 \text{ a.e.}\}.$$

Thus $L^1(X;\mathbb{C})$ consists of equivalence classes $[f]$ where $f \sim g$ provided $f = g$ almost everywhere. However, it is customary to confuse an integrable function with its equivalence class and drop the [] from the notation.

In light of Proposition 5.1 we obtain

Proposition 7.1. *$L^1(X;\mathbb{C})$ is a normed space with respect to the norm $\|f\|_1 = \int |f|\, d\mu$.*

In general, for $1 \le p < \infty$, we say a measurable function $f : X \longrightarrow \mathbb{C}$ is ***p*-integrable** if

$$\int |f|^p \, d\mu < \infty.$$

As above, finiteness of this integral implies that $\int f\, d\mu \in \mathbb{C}$ exists, and $\int |f|^p \, d\mu = 0$ if and only if $f = 0$ a.e. We define

$$L^p(X;\mathbb{C}) = \{f : X \longrightarrow \mathbb{C} \ p\text{-integrable}\}/\mathcal{Z}.$$

The proofs of the following two results are essentially the same as for the sequence spaces ℓ^p[2]:

Proposition 7.2 (Hölder's inequality)**.** *Let $1 < p < \infty$ and $1/p + 1/q = 1$, and $f \in L^p(X;\mathbb{C})$, $g \in L^q(X;\mathbb{C})$. Then $fg \in L^1(X;\mathbb{C})$ and*

$$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

Proposition 7.3 (Minkowski's inequality)**.** *Let $f, g \in L^p(X;\mathbb{C})$, $1 \le p < \infty$. Then $f + g \in L^p(X;\mathbb{C})$ and*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p$$

Corollary 7.4. *The spaces $L^p(X;\mathbb{C})$, for $1 \le p < \infty$ are normed vector spaces.*

To show that $L^p(X;\mathbb{C})$ is complete, and hence a Banach space, it is convenient to use the next result, which gives an alternate characterization of completeness for normed spaces.

A series $\sum_{n=1}^{\infty} x_n$ in a normed space $(X, \|\cdot\|)$ is said to **converge** if the sequence $s_k = \sum_{n=1}^{k} x_n$ of partial sums converges to some $s \in X$, and then we write $s = \sum_{n=1}^{\infty} x_n$. The series is said to be **absolutely convergent** if the series $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$.

Proposition 7.5. *A normed space $(X, \|\cdot\|)$ is complete if and only if every absolutely convergent series converges.*

[2] Observe that ℓ^p is precisely $L^p(\mathbb{N};\mathbb{C})$ when $\mathbb{N}$ is equipped with the **counting measure** $m : S \subseteq \mathbb{N} \longrightarrow |S|$.

Proof. Suppose X is complete and $\sum_{n=1}^{\infty} \|x_n\|$ converges. Then

$$\|s_k - s_m\| = \Big\| \sum_{n=k}^{m} x_n \Big\| \leq \sum_{n=k}^{m} \|x_n\|$$

so (s_k) is Cauchy and hence convergent.

Conversely, suppose every absolutely convergent series converges in X, and let (x_n) be a Cauchy sequence. Define a subsequence (x_{n_k}) by the condition

$$\|x_i - x_j\| \leq 2^{-k} \quad \forall i, j \geq n_k.$$

If we set $y_1 = x_{n_1}$ and $y_k = x_{n_k} - x_{n_{k-1}}$ for $k > 1$, then we may express the x_{n_k} as partial sums $x_{n_k} = \sum_{i=1}^{k} y_i$. Since

$$\sum_{i=1}^{\infty} \|y_i\| = \sum_{i=1}^{\infty} \big\|x_{n_i} - x_{n_{i-1}}\big\| \leq \sum_{i=1}^{\infty} 2^{1-i} < \infty,$$

it follows by hypothesis that $\lim_{k\to\infty} x_{n_k} = \sum_{i=1}^{\infty} y_i$ exists. Since (x_n) is Cauchy and converges along a subsequence, (x_n) itself converges to the same limit. □

Proposition 7.6. *$L^p(X; \mathbb{C})$ is complete.*

Proof. Let (f_n) be a sequence in $L^p(X;\mathbb{C})$ such that $\sum_{n=1}^{\infty} \|f_n\|_p = B < \infty$ converges in $\mathbb{R}$. By the previous result, it suffices to show that $\sum_{n=1}^{\infty} f_n$ converges in $L^p(X;\mathbb{C})$.

Set $g_k = \sum_{n=1}^{k} |f_n|$. Then (g_k) is a sequence of positive functions which is pointwise increasing, hence converges pointwise to $g = \sum_{n=1}^{\infty} |f_n| \in \mathcal{L}^+$, and $g_k^p \to g^p \in \mathcal{L}^+$ as a pointwise increasing sequence as well. (Note that g may take the value ∞.) The norms $\|g_k\|_p$ are uniformly bounded:

$$\|g_k\|_p = \Big\| \sum_{n=1}^{k} |f_n| \Big\|_p \leq \sum_{n=1}^{k} \|f_n\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p = B,$$

and then by the Monotone Convergence Theorem,

$$\|g\|_p^p = \int g^p \, d\mu = \lim_k \int g_k^p \, d\mu = \lim_k \|g_k\|_p^p \leq B,$$

so $g \in L^p\big(X; [0,\infty)\big)$. In particular $g(x) = \sum_{n=1}^{\infty} |f_n(x)|$ is finite almost everywhere.

Now consider the series $\sum_{n=1}^{\infty} f_n$. Since $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely almost everywhere, it follows that it converges a.e. Writing $f = \sum_{n=1}^{\infty} f_n$ for this a.e. limit, it follows from the fact that $|f| \leq g$ a.e. and $g \in L^p(X;\mathbb{R})$ that $f \in L^p(X;\mathbb{C})$.

Finally, to show that the series converges in the L^p norm, observe that

$$\Big| f - \sum_{n=1}^{k} f_n \Big|^p \leq \Big| |f| + \sum_{n=1}^{k} |f_n| \Big|^p \leq 2^p g^p \in L^1(X;\mathbb{C}),$$

and then by the Dominated Convergence Theorem it follows that

$$\lim_{k\to\infty} \Big\| f - \sum_{n=1}^{k} f_n \Big\|_p^p = \lim_{k\to\infty} \int \Big| f - \sum_{n=1}^{k} f_n \Big|^p d\mu = \int \lim_{k\to\infty} \Big| f - \sum_{n=1}^{k} f_n \Big|^p d\mu = 0$$

hence the series converges in $L^p(X;\mathbb{C})$. □

The isomorphism $(\ell^p)' \cong \ell^q$ for $1/p+1/q=1$ has a natural analogue for $L^p(X;\mathbb{C})$. The proof, which uses the Radon-Nikodym theorem, is outside the scope of these short notes.

Theorem 7.7. *Let $1<p<\infty$ and $1/p+1/q=1$. Then the map*

$$L^q(X;\mathbb{C}) \ni g \longmapsto F_g \in \big(L^p(X;\mathbb{C})\big)', \quad F_g(f) = \int gf\,d\mu$$

is an isometry.

What about $(L^1)'$? It turns out that the natural analogue of ℓ^∞ is the space $L^\infty(X;\mathbb{C})$ of (a.e. equivalence classes of) measurable functions $f : X \longrightarrow \mathbb{C}$ which are bounded almost everywhere. This space is equipped with the norm

$$\|f\|_\infty = \inf\{M : \mu(\{f > M\}) = 0\} = \inf\left\{\sup_x |g(x)| : g = f \text{ a.e.}\right\},$$

with respect to which $(L^\infty(X;\mathbb{C}), \|\cdot\|_\infty)$ may be shown to be a Banach space. Under some conditions on $(X,\mathcal{A},\mu)$—in particular if it is **σ-finite**, meaning $X = \bigcup_{n=1}^\infty E_n$ with $\mu(E_n) < \infty$ which holds in particular for $\mathbb{R}^n$ with Lebesgue measure—then the map

$$L^\infty(X;\mathbb{C}) \ni g \longmapsto F_g \in \big(L^1(X;\mathbb{C})\big)', \quad F_g(f) = \int gf\,d\mu$$

is again an isometry. There is *almost never* an isometry between L^1 and $(L^\infty)'$, except in very limited cases, such as when X is a finite set with counting measure; in this case $L^p(X)$ is simply $\mathbb{C}^n$, $n=|X|$, for each p, with $\|x\|_p = \big(\sum_{i=1}^n x_i^p\big)^{1/p}$, and $\|x\|_\infty = \max\{|x_1|,\dots,|x_n|\}$.