Bade

Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													

Math 2321 Final Exam

May 1, 2015

Instructor's name____________________ Your name____________________

Please check that you have 10 different pages.

Answers from your calculator, without supporting work, are worth zero points.

1) Consider the function $f(x, y) = xe^{xy} + y\sin(x)$. Let $\mathbf{p}$ be the point $(2, 0)$, and let $\mathbf{v}$ be the vector from the point $(0, 1)$ to the point $(3, 5)$.

a) (5 points) Find the directional derivative $D_{\mathbf{u}}f(\mathbf{p})$, where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{v}$.

$\underline{v} = (3,5)-(0,1) = (3,4)$. $\underline{u} = \frac{(3,4)}{|(3,4)|} = \frac{1}{5}(3,4)$.

$\vec{\nabla} f = (xe^{xy}y + e^{xy} + y\cos x,\ x^2e^{xy} + \sin x)$.

$\vec{\nabla} f(2,0) = (0 + e^0 + 0,\ 4e^0 + \sin 2) = (1, 4+\sin 2)$.

$D_{\underline{u}} f(p) = (1, 4+\sin 2) \cdot \frac{1}{5}(3,4)$

$= \frac{1}{5}(3 + 16 + 4\sin 2) = \boxed{\frac{1}{5}(19 + 4\sin 2)}$.

b) (3 points) Find the direction, as a unit vector, in which f increases most rapidly at $\mathbf{p}$.

$$\frac{\vec{\nabla} f(2,0)}{|\vec{\nabla} f(2,0)|} = \boxed{\frac{(1, 4+\sin 2)}{\sqrt{1 + (4+\sin 2)^2}}}.$$

Bade

2) (3 points each) Consider the function $f(x, y, z) = xe^y + xz^2$

a) Find the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ of the function f at the point $\mathbf{p} = (2, 0, 1)$.

$$\frac{\partial f}{\partial x} = e^y + z^2.$$

$$\frac{\partial f}{\partial y} = xe^y.$$

$$\frac{\partial f}{\partial z} = 2xz.$$

At $(2,0,1)$,

$$\boxed{\begin{aligned} f_x &= \frac{\partial f}{\partial x} = e^0 + 1^2 = 2, \\ f_y &= \frac{\partial f}{\partial y} = 2e^0 = 2, \text{ and} \\ f_z &= \frac{\partial f}{\partial z} = 2\cdot 2\cdot 1 = 4. \end{aligned}}$$

b) Find the linearization $L(x, y, z)$ of $f(x, y, z)$ at $\mathbf{p}$.

$$L(x,y,z) = f(2,0,1) + f_x(2,0,1)(x-2) + f_y(2,0,1)(y-0) + f_z(2,0,1)(z-1).$$

$$\boxed{f(2,0,1) = 2e^0 + 2\cdot 1^2 = 4.}$$

$$\boxed{L(x,y,z) = 4 + 2(x-2) + 2(y-0) + 4(z-1).}$$

c) Use the linearization of f to estimate the value of f at $(1.9, 0.1, 1.5)$. (The "exact" value of $f(1.9, 0.1, 1.5)$ from your calculator is worth zero points.)

$$f(1.9, 0.1, 1.5) \approx L(1.9, 0.1, 1.5) =$$

$$4 + 2(1.9-2) + 2(0.1) + 4(1.5-1) =$$

$$4 - 0.2 + 0.2 + 2 = \boxed{6.}$$

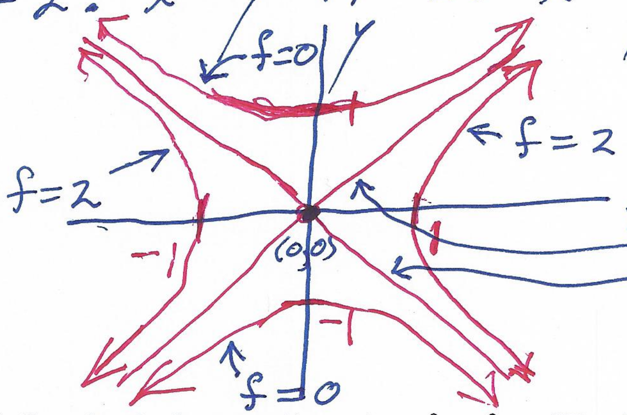
Pichler

3) (7 points) Consider the function $f(x, y) = x^2 - y^2 + 1$. On a single copy of the xy-plane, sketch the levels curves, where $f = 0$, $f = 1$, and $f = 2$. In your sketch, indicate which level curves are which, and include the value all of the intercepts with the $x-$ and $y-$axes.

$f=0: x^2-y^2+1=0.\ y^2-x^2=1.$

$f=1: x^2-y^2+1=1.\ x^2=y^2.\ x=\pm y.$

$f=2: x^2-y^2+1=2.\ x^2-y^2=1.$



4) Consider the function $f(x, y, z) = x^2 + y^2 + \sqrt{xz}$. $= x^2+y^2+x^{\frac{1}{2}}z^{\frac{1}{2}}.$

a) (7 points) Find an equation of the tangent plane to the level surface of f at the point $(1, -1, 1)$.

$\vec{\nabla} f = \left(2x+\frac{1}{2}x^{-\frac{1}{2}}z^{\frac{1}{2}},\ 2y,\ \frac{1}{2}x^{\frac{1}{2}}z^{-\frac{1}{2}}\right).$

$\vec{\nabla} f(1,-1,1) = \left(2+\frac{1}{2},\ -2,\ \frac{1}{2}\right) = \left(\frac{5}{2}, -2, \frac{1}{2}\right).$

Tangent plane:

$$\boxed{\frac{5}{2}(x-1)-2(y--1)+\frac{1}{2}(z-1)=0.}$$

b) (3 points) Find the coordinates of the point of intersection of the x-axis with the tangent plane from part (a).

x-axis is where $y=z=0.$

$\frac{5}{2}(x-1)-2(1)-\frac{1}{2}=0.\quad \frac{5}{2}(x-1)=\frac{5}{2}.$

$x-1=1.\ \boxed{x=2.}$

$$\boxed{(x,y,z)=(2,0,0).}$$

Hepler

5) (10 points) Find the points at which the function $f(x, y) = 2x^2 + y^2 + 2x^2y$ attains a local minimum value, a local maximum value, or has a saddle point.

$f_x = 4x + 4xy = 0.$ $4x(1+y) = 0.$ $x = 0$ or $y = -1.$

$f_y = 2y + 2x^2 = 0.$ $y + x^2 = 0.$ $x = 0$ and $y = 0$ or $y = -1.$ $x^2 = 1.$ $x = \pm 1.$

Three critical points:

$(x, y) = (0,0), (1,-1), (-1,-1).$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4+4y & 4x \\ 4x & 2 \end{vmatrix} = 8 + 8y - 16x^2.$$

$D = 8[1 + y - 2x^2].$

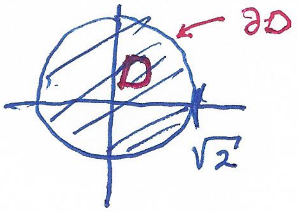
At $(0,0)$: $D = 8 > 0.$ $f_{xx} = 4 > 0.$ local min.

At $(1,-1)$: $D = 8[1-1-2] = -16 < 0.$ saddle pt.

At $(-1,-1)$: $D = 8[1-1-2] = -16 < 0.$ saddle pt.

Topalov

6) (10 points) Find the global minimum and the global maximum values of the function $f(x, y) = -2 + x^2 + 2y^2$ in the closed disk D where $x^2 + y^2 \le 2$, as well as the coordinates of the points where these extreme values are attained.



Find interior crit. pts.:

$f_x = 2x = 0.$ $f_y = 4y = 0.$ $(x, y) = (0, 0).$

Find critical points of $f|_{\partial D}$:

On boundary, $x^2 + y^2 = 2$ and $-\sqrt{2} \le y \le \sqrt{2}$.

$f_{\partial D}(y) = -2 + (x^2 + y^2) + y^2 = -2 + 2 + y^2 = y^2.$

Check where $f'_{\partial D}(y) = 0$, i.e., $2y = 0$. $y = 0$. $x^2 = 2$.

Two new points to check: $(x, y) = (\pm\sqrt{2}, 0).$

Check endpoints on interval, where $y = \pm\sqrt{2}$ and $x = 0$.

Two more points: $(x, y) = (0, \pm\sqrt{2}).$

(x, y)	$-2 + x^2 + 2y^2$
$(0, 0)$	-2
$(-\sqrt{2}, 0)$	0
$(\sqrt{2}, 0)$	0
$(0, -\sqrt{2})$	2
$(0, \sqrt{2})$	2

Minimum value is -2, at $(0, 0)$.

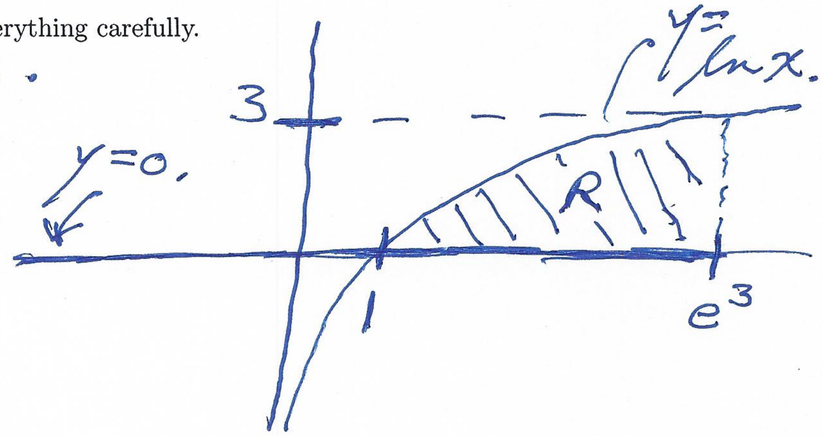
Maximum value is 2, at $(0, -\sqrt{2})$ and $(0, \sqrt{2})$.

Veliche

7) Consider a double integral, which is equal to the given iterated integral: $\int\int_R (x^3 - y^2)\, dA = \int_1^{e^3} \int_0^{\ln x} (x^3 - y^2)\, dy\, dx.$

a) (3 points) Sketch the region of integration R. Label everything carefully.

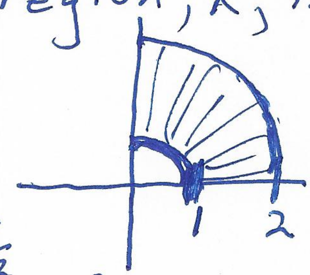
$1 \le x \le e^3. \quad 0 \le y \le \ln x.$



b) (5 points) Reverse the order of integration from what you were originally given, write down the resulting iterated integral using the new order. DO NOT EVALUATE THIS INTEGRAL.

$$\int_0^3 \int_{e^y}^{e^3} (x^3 - y^2)\, dx\, dy$$

8) (10 points) A solid occupies the region S, which is in the 1st octant (where $x \ge 0, y \ge 0$, and $z \ge 0$) of the solid where $0 \le z \le 1$ and $1 \le x^2 + y^2 \le 4$. Suppose that x, y, z are measured in meters, and the solid has density given by $\delta(x, y, z) = \frac{5}{x^2+y^2}$ kg/m^3. Calculate the mass of the solid. Projected region, R, in xy-plane:



$$\text{Mass} = \iiint_S \delta\, dV$$

$$= \int_0^{\pi/2} \int_1^2 \int_0^1 \frac{5}{r^2}\, r\, dz\, dr\, d\theta = \int_0^{\pi/2} \int_1^2 \frac{5}{r}\, dr\, d\theta =$$

$$= \int_0^{\pi/2} 5 \ln r \Big|_1^2 d\theta = \int_0^{\pi/2} 5 \ln 2\, d\theta =$$

$$\boxed{\frac{5(\ln 2)\pi}{2} \text{ kg.}}$$

Nguyen

9) This problem concerns the vector field $\mathbf{F}(x, y, z) = (ye^z,\ xe^z + z,\ xye^z + y + 2z)$.

a) (2 points) Show that $\mathbf{F}$ is conservative, **without** producing a potential function for $\mathbf{F}$. Defined on all of $\mathbb{R}^3$, which is simply-connected.

$$\vec{\nabla} \times \underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z + z & xye^z + y + 2z \end{vmatrix} =$$

$$(xe^z + 1 - (xe^z + 1), -(ye^z - ye^z), e^z - e^z) = (0, 0, 0).$$

Thus, $\underline{F}$ is conservative.

b) (4 points) Find a potential function $f(x, y, z)$ for $\mathbf{F}(x, y, z)$.

Want: $\frac{\partial f}{\partial x} = ye^z$. $\frac{\partial f}{\partial y} = xe^z + z$. $\frac{\partial f}{\partial z} = xye^z + y + 2z$.

$f = \int ye^z\,dx = xye^z + A(y, z)$.

$\frac{\partial f}{\partial y} = xe^z + z = xe^z + \frac{\partial A}{\partial y}$. So, $\frac{\partial A}{\partial y} = z$. $A(y, z) = \int z\,dy$
$= yz + B(z)$.

Now have: $f = xye^z + yz + B(z)$.

$\frac{\partial f}{\partial z} = xye^z + y + 2z = xye^z + y + B'(z)$.

$B'(z) = 2z$. $B(z) = z^2 + C$.

A potential function:

$$\boxed{f = xye^z + yz + z^2.}$$

c) (2 points) Compute the line integral $\int_C \mathbf{F} \cdot dr$, where C is the curve in $\mathbb{R}^3$ consisting of straight line segments from $(0, 0, 0)$ to $(1, 1, 0)$, then to $(2, 1, 1)$, and finally to $(3, 2, 1)$.

Fund. Thm. of Line Integrals:

$$\int_C \underline{F} \cdot d\underline{r} = f(3, 2, 1) - f(0, 0, 0) = 6e + 2 + 1 - 0$$
$$= \boxed{6e + 3.}$$

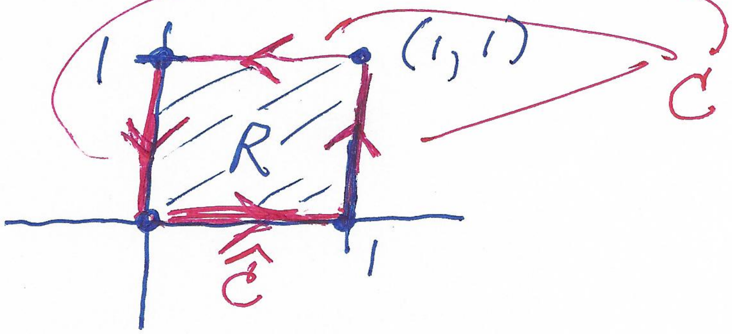
Massey

10) Suppose $\mathbf{F} = (\overset{P}{3y + x^2},\ \overset{Q}{x^2 + e^{-y^2}})$ represents a force field in Newtons, where x and y are in meters.

a) (1 point) Compute the 2-dimensional curl of $\mathbf{F}$.

$$Q_x - P_y = 2x - 3.$$

b) (5 points) Find the work done by $\mathbf{F}$ on a particle which moves along the curve C given by three sides of a square, starting from $(1, 0)$, to $(1, 1)$, then to $(0, 1)$, and finally to $(0, 0)$.



Use Green's Theorem.

$$\int_C \underline{F}\cdot d\underline{r} = \int_{C+\hat{C}} \underline{F}\cdot d\underline{r} - \int_{\hat{C}} \underline{F}\cdot d\underline{r} =$$

$$\iint_R (Q_x - P_y)\,dA - \int_{\hat{C}} P\,dx + Q\,dy =$$

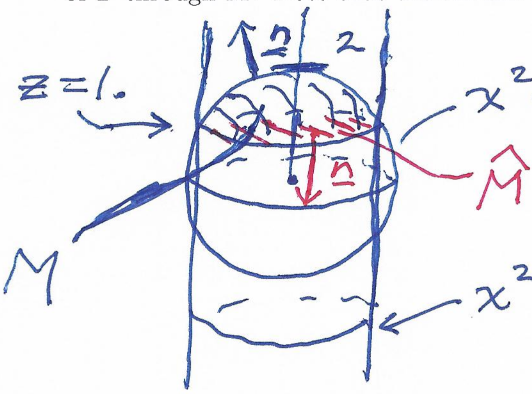
$$\int_0^1\int_0^1 (2x-3)\,dy\,dx - \int_0^1 \underbrace{x^2}_{P(x,0)}\,dx$$

$$= \int_0^1 (2x-3)\,dx - \frac{x^3}{3}\Big|_0^1 = x^2 - 3x\Big|_0^1 - \frac{1}{3}$$

$$= 1 - 3 - \frac{1}{3} = \boxed{-\frac{7}{3}\text{ joules.}}$$

Kottke

11) (7 points) Consider the vector field $\mathbf{F}(x, y, z) = (\cos z + y^2 e^z, xe^{-z}, z^2)$. Let M be the part of the hemisphere where $z = \sqrt{4 - x^2 - y^2}$ that lies inside the cylinder where $x^2 + y^2 = 3$, oriented upward. Compute the flux integral $\displaystyle\iint_M \mathbf{F} \cdot \mathbf{n}\, dS$ of $\mathbf{F}$ through M. Note that the surface M is **not** closed.



$x^2+y^2+z^2=4.$

$x^2+y^2=3.$

Use Divergence Thm.

Add closing disk, $\hat{M}$, where $z=1$ and $x^2+y^2 \le 3$, oriented downward.

$$\iint_M \underline{F} \cdot \underline{n}\, dS = \iint_{M \cup \hat{M}} \underline{F} \cdot \underline{n}\, dS - \iint_{\hat{M}} \underline{F} \cdot \underline{n}\, dS =$$

$$\iiint_E (0+0+2z)\,dV - \iint_{\hat{M}} (*, *, \underset{z=1}{z^2}) \cdot (0,0,-1)\, \underset{\text{in } xy\text{-plane}}{dA}$$

E: enclosed solid region

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} 2z\, r\, dz\, dr\, d\theta \underbrace{- \iint_M (-1)\, dA}_{\text{area of disk}}$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(r z^2 \Big|_{z=1}^{z=\sqrt{4-r^2}} \right) dr\, d\theta + 3\pi =$$

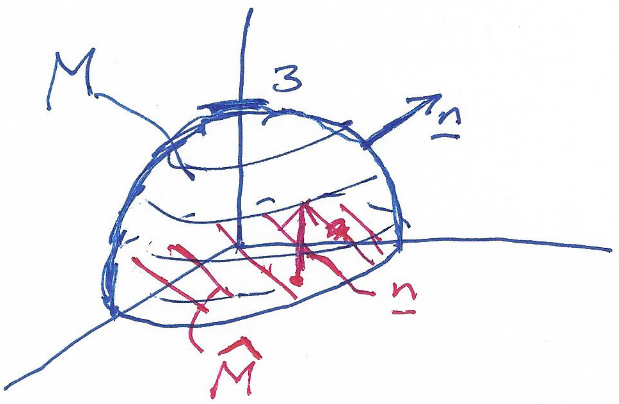
$$\int_0^{2\pi}\int_0^{\sqrt{3}} \left(r(4-r^2) - r \right) dr\, d\theta + 3\pi = \int_0^{2\pi}\int_0^{\sqrt{3}} (3r - r^3)\, dr\, d\theta + 3\pi =$$

$$2\pi \left(\frac{3r^2}{2} - \frac{r^4}{4} \right)\Big|_0^{\sqrt{3}} + 3\pi$$

$$= 2\pi \left(\frac{9}{2} - \frac{9}{4} \right) + 3\pi = \left(\frac{9}{2} + 3 \right)\pi = \boxed{\frac{15}{2}\pi.}$$

Massey

12) (7 points) Consider the vector field $\mathbf{F}(x, y, z) = (2y\cos z\,,\ e^x \sin z\,,\ e^z)$. Let M be the **top hemisphere** of the sphere of radius 3, centered at the origin, oriented upward. Compute the flux integral $\int\int_M (\vec{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\, dS$.



By Stokes'* Thm.,

$$\iint_M (\vec{\nabla} \times \underline{F}) \cdot \underline{n}\, dS =$$

$$\iint_{\hat{M}} (\vec{\nabla} \times \underline{F}) \cdot \underline{n}\, dS = \iint_{\hat{M}} (\vec{\nabla} \times \underline{F}) \cdot (0,0,1)\, \underbrace{dA}_{\text{in } xy\text{-plane}}$$

$$= \iint_{\hat{M}} (e^x \sin z - 2\cos z)\, dA =$$

$\hat{M} \leftarrow z = 0$

$$\iint_{\hat{M}} -2\, dA = -2\left(\text{area of } \hat{M}\right) = -2\pi(3)^2$$

$$= \boxed{-18\pi.}$$

$$\vec{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y\cos z & e^x \sin z & e^z \end{vmatrix}$$

$$= (*, *, e^x \sin z - 2\cos z).$$