MATH 3150 – HOMEWORK 5. DUE 10/16

Update 10/9: Added #7, 8 and 9, as well as parts (d)–(f) of #3 and part (i) of #6.

Problem 1 (p. 108, #4)**.** Let $B \subset \mathbb{R}^n$ be any set. Define $C = \{x \in \mathbb{R}^n \mid d(x, y) < 1 \text{ for some } y \in B\}$. Show that C is open.

Problem 2 (p. 109, #1)**.** Find the interior of the set $R = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 \leq 1\}$.

Problem 3 (p. 145, #12, #14)**.** Prove the following properties for subsets A and B of a metric space:
(a) $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$.
(b) $\operatorname{int}(A \cup B) \supset \operatorname{int}(A) \cup \operatorname{int}(B)$.
(c) $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$.
(d) $\operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(A)$.
(e) $\operatorname{cl}(A \cap B) \supset \operatorname{cl}(A) \cap \operatorname{cl}(B)$.
(f) $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$.

Problem 4 (p. 108, #6)**.** Show that $\mathbb{R}^2$ with the taxicab metric $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$ has the same open sets as it does with the standard metric. (Hint: think about the shapes of the balls with respect to each metric.)

Problem 5 (p. 146, #23)**.** Let (M, d) be a metric space. Prove that the interior of a set $A \subset M$ is the union of all the subsets of A that are open.

Problem 6 (p. 143, #1, #2)**.**

- (i) Determine whether the following sets are open or closed:
  - (a) $(1, 2)$ in $\mathbb{R}^1 = \mathbb{R}$
  - (b) $[2, 3]$ in $\mathbb{R}$
  - (c) $\bigcap_{n=1}^{\infty} [-1, 1/n)$ in $\mathbb{R}$
  - (d) $\mathbb{R}^n$ in $\mathbb{R}^n$
  - (e) $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$
  - (f) $\{r \in (0, 1) \mid r \text{ is rational}\}$ in $\mathbb{R}$
  - (g) $\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1\}$ in $\mathbb{R}^2$
  - (h) $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ in $\mathbb{R}^n$
  - (i) $\{x_k \in \mathbb{R}^n\}$ for a sequence x_k in $\mathbb{R}^n$.
- (ii) Determine the interiors, closures and boundaries of the sets in (i).

Problem 7 (p. 120, #4)**.** Is it always true that $\operatorname{bd}(A) = \operatorname{bd}(\operatorname{int}(A))$? Why or why not?

Problem 8 (p. 147, #34)**.** Let $x_k \in \mathbb{R}^n$ be a sequence such that $d(x_{k+1}, x_k) \leq r d(x_k, x_{k-1})$ for some $0 \leq r < 1$. Show that x_k converges.

Problem 9 (p. 149, #53)**.** Given a set A in a metric space (M, d), what is the maximum number of distinct subsets of M that can be produced by successively applying the operations of closure, interior and complement to A (in any order)? Give an example of a set A which achieves this maximum number.