MATH 3150 – HOMEWORK 6

Updated 10/28: added Problem 5.

Problem 1 (p. 155 #1)**.** Show that $A \subset M$ is sequentially compact if and only if every infinite subset of A has an accumulation point in A.

Problem 2 (p. 155 #2, p. 173 #5)**.** Show that the following sets are not compact:
(a) $\{(x, y) \in \mathbb{R}^2 \mid 0 \le x < 1,\ 0 \le y \le 1\}$
(b) $\{x \in \mathbb{R}^n \mid \|x\| < 1\}$
(c) $\mathbb{Z}$, the set of integers in $\mathbb{R}$

Problem 3 (p. 156 #2, p. 175 #25)**.**
(a) Let $r_1, r_2, r_3, \dots$ be an enumeration of the rational numbers in the interval $[0, 1]$. Show that there is a convergent subsequence.
(b) Prove that there is a sequence of distinct integers $n_1, n_2, \dots \longrightarrow \infty$ such that $\lim_{k\to\infty} \sin(n_k)$ exists.

Problem 4 (p. 173 #8)**.** Let $A \subset \mathbb{R}^n$ be a compact set and let x_k be a Cauchy sequence in $\mathbb{R}^n$ with $x_k \in A$. Show that x_k converges to a point in A.

Problem 5. Let (M_1, d_1) and (M_2, d_2) be metric spaces with compact sets $K_1 \subset M_1$ and $K_2 \subset M_2$. Show that $K_1 \times K_2$ is a compact subset of the space $(M_1 \times M_2, d = d_1 + d_2)$. (The metric d on the product $M_1 \times M_2$ is defined by $d\big((x_1, x_2), (y_1, y_2)\big) = d_1(x_1, y_1) + d_2(x_2, y_2)$.)