AN INTRODUCTION TO L^2-COHOMOLOGY

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Introduction

Lecture notes for a minicourse at "London Summer School and Workshop: The Sen Conjecture and Beyond", June 19 – June 23, 2017, at UCL.

1. First lecture: Generalities

Reference recommendation for an overview:

- Gilles Carron, 'L^2 harmonic forms on non-compact manifolds',
- Xianzhe Dai, 'An introduction to L^2 cohomology',
- Steven Zucker, 'L_2 Cohomology of Warped Products and Arithmetic Groups', sections 1 and 2,
- Leslie Saper, 'L_2-cohomology of algebraic varieties',
- Daniel Grieser and Matthias Lesch, 'On the L^2-Stokes theorem and Hodge theory for singular algebraic varieties'

1.1. **Topological manifolds.** For fun let us start by recalling that there is an analytic approach to cohomology on most *topological* manifolds[1]. Indeed, let M be a compact, oriented, connected topological manifold without boundary of dimension $m \neq 4$. A theorem of Sullivan guarantees that M has an (essentially unique) Lipschitz structure. That is, we can parametrize M with charts homeomorphic to the unit ball in $\mathbb{R}^m$ so that the transition functions are locally bi-Lipschitz.

Lipschitz functions are differentiable almost everywhere with derivatives in L^∞ so bi-Lipschitz homeomorphisms preserve the class of Lebesgue measure and a Lipschitz manifold has a canonical measure class and a well-defined set of L^2-functions. Since M is not smooth we can not talk about a cotangent bundle but we can still make sense of L^2 differential form of degree j,

$$\omega \in L^2\Omega^j(M),$$

as an object that in each coordinate chart is a differential form on $\mathbb{R}^m$,

$$\sum f_{i_1,\ldots,i_j}\, dx^{i_1} \wedge \ldots \wedge dx^{i_j},$$

with coefficient functions in L^2. The usual transformation rules are assumed to hold and they preserve the L^2 condition as they involve multiplication by determinants of matrices of (weak) partial derivatives of transition functions, and these are all in L^∞.

Similarly a Riemannian metric is defined without reference to vector bundles as a compatible collection of measurable Riemannian metrics, one on each coordinate chart, with the

[1]We follow the exposition in "Applications of Analysis on Lipschitz manifolds" by Jonathan Rosenberg where one can find references to the papers of Sullivan and Teleman in which these results were established.

property that they define L^2-norms equivalent to the L^2-norm on $\mathbb{R}^m$. As in the smooth setting this gives rise to an inner product on L^2 differential forms,

$$\langle \omega, \phi \rangle = \int_M \omega \wedge * \overline{\beta},$$

which gives $L^2\Omega^j(M)$ the structure of a Hilbert space.

Given $\omega \in L^2\Omega^j(M)$ and $\phi \in L^2\Omega^{j+1}(M)$, we say that $\phi = d\omega$ (weakly) if in each coordinate chart, $\mathcal{U}$, for every smooth differential form of compact support, $\beta \in \mathcal{C}_c^\infty \Omega^{m-(j+1)}(\mathcal{U})$, we have

$$\int_{\mathbb{R}^m} \omega \wedge d\beta = (-1)^{j+1} \int_{\mathbb{R}^m} \phi \wedge \beta.$$

This relation is unchanged by coordinate changes because the usual relation $f^* d\alpha = d(f^*\alpha)$ holds when f is a Lipschitz map between relatively compact open subsets of $\mathbb{R}^m$ and the coefficients of α and $d\alpha$ are measurable functions[2]. It is easy an easy exercise to see that the square of the exterior derivative, computed weakly, vanishes.

Let

$$\mathscr{D}_{\max}(d)_j = \{\omega \in L^2\Omega^j(M) : d\omega \in L^2\Omega^{j+1}(M)\}$$

denote the L^2 differential forms of degree j whose exterior derivative is also an L^2 differential form. We have a complex of vector spaces,

$$0 \longrightarrow \mathscr{D}_{\max}(d)_0 \xrightarrow{d} \mathscr{D}_{\max}(d)_1 \xrightarrow{d} \dots \xrightarrow{d} \mathscr{D}_{\max}(d)_m \longrightarrow 0,$$

and corresponding homology groups

$$\mathrm{H}^j(\mathscr{D}_{\max}(d)_\bullet, d) = \frac{\mathrm{Ker}\left(\mathscr{D}_{\max}(d)_j \xrightarrow{d} \mathscr{D}_{\max}(d)_{j+1}\right)}{\mathrm{Im}\left(\mathscr{D}_{\max}(d)_{j-1} \xrightarrow{d} \mathscr{D}_{\max}(d)_j\right)}.$$

To identify these groups with the singular cohomology of M it is convenient to use sheaves. We define a pre-sheaf on M by assigning to each open set $\mathcal{U} \subseteq M$ the vector space

$$\mathscr{D}_{\max}(d; \mathcal{U}) = \{\omega \in L^2\Omega^j(\mathcal{U}) : d\omega \in L^2\Omega^{j+1}(\mathcal{U})\}$$

and assigning to each inclusion $j : \mathcal{V} \hookrightarrow \mathcal{U}$ of open sets the restriction map

$$j^* : \mathscr{D}_{\max}(d; \mathcal{U}) \longrightarrow \mathscr{D}_{\max}(d; \mathcal{V})$$

This presheaf is filtered by differential form degree and the exterior derivative makes it into a complex of presheaves. We sheafify and denote the corresponding sheaf complex by $\mathbf{L}^2_{\max}\mathbf{\Omega}^\bullet$. Since we have a Poincaré Lemma[3] the complex of sheaves

$$0 \longrightarrow \mathbb{R}_M \longrightarrow \mathbf{L}^2_{\max}\mathbf{\Omega}^0 \longrightarrow \mathbf{L}^2_{\max}\mathbf{\Omega}^1 \longrightarrow \dots$$

is exact and is a soft resolution of the constant sheaf $\mathbb{R}_M$. It follows that the sheaf cohomology is the singular cohomology with $\mathbb{R}$ coefficients.

[2] This is established in Theorem 9C of Geometric Integration Theory by Hassler Whitney.

[3] As explained below, on a smooth manifold the L^2-cohomology can be computed using a subcomplex of smooth differential forms. Thus if $\mathcal{U}$ is a coordinate chart we can compute the L^2-cohomology of $\mathcal{U}$ using smooth differential forms, and these satisfy the usual Poincare Lemma.

1.2. **Hilbert complexes.** This mix of homological algebra and functional analysis is usefully framed as follows[4].

Definition 1. *Let $N \in \mathbb{N}$. A* **Hilbert complex** *of length N consists of Hilbert spaces, H_j, $j \in \{0, \dots, N\}$, with $H_{N+1} = \{0\}$, closed[5], densely defined operators*

$$D_j : H_j \longrightarrow H_{j+1}, \quad j \in \{0, \dots, N-1\},$$

with domain $\mathcal{D}_j = \mathscr{D}(D_j)$ and range $\mathcal{R}_j \subseteq H_{j+1}$ satisfying

$$\mathcal{R}_j \subseteq \mathcal{D}_{j+1} \text{ and } D_{j+1} \circ D_j = 0.$$

We denote this by

$$0 \longrightarrow \mathcal{D}_0 \xrightarrow{D_0} \mathcal{D}_1 \xrightarrow{D_1} \dots \xrightarrow{D_{N-1}} \mathcal{D}_N \longrightarrow 0$$

when the rest of the data are understood.

If the ranges $\mathcal{R}_j$ are all closed and the homology groups

$$\mathrm{H}_j((\mathcal{D}_\bullet, D_\bullet)) = \frac{\operatorname{Ker}\left(\mathcal{D}_j \xrightarrow{D_j} \mathcal{D}_{j+1}\right)}{\operatorname{Im}\left(\mathcal{D}_{j-1} \xrightarrow{D_{j-1}} \mathcal{D}_j\right)}$$

are all finite dimensional, we say that the Hilbert complex is a **Fredholm complex***.*

Every Riemannian manifold, (M, g), compact or not, gives rise to a Hilbert complex with $H_j = L^2\Omega^j(M)$, $D_j = d$ and

$$\mathcal{D}_j = \{\omega \in L^2\Omega^j(M) : d\omega \in L^2\Omega^{j+1}(M)\} = \mathscr{D}_{\max}(d)_j$$

known as the *maximal domain* of d. The cohomology of this Hilbert complex is the **L^2-cohomology of** (M, g) and is denoted

$$\mathrm{H}^j_{L^2}(M, g).$$

We also define the **reduced L^2-cohomology of** (M, g) to be the groups

$$\overline{\mathrm{H}}^j_{L^2}(M, g) = \frac{\operatorname{Ker}\left(\mathscr{D}_{\max}(d)_j \xrightarrow{d} \mathscr{D}_{\max}(d)_{j+1}\right)}{\overline{\operatorname{Im}\left(\mathscr{D}_{\max}(d)_{j-1} \xrightarrow{d} \mathscr{D}_{\max}(d)_j\right)}}$$

obtained as the quotient of the kernel of d by the *closure* of the image of d. While this is not the cohomology of a Hilbert complex, it is closely related to harmonic forms as we will soon see. Moreover if the L^2-cohomology does not coincide with the reduced L^2-cohomology then the former is necessarily infinite dimensional.

If (M, g) is a smooth Riemannian manifold, as we assume from now on, it has another canonical Hilbert complex. The **minimal closed extension of** d is the exterior derivative endowed with the domains

$$\mathscr{D}_{\min}(d)_j = \{\omega \in L^2\Omega^j(M) : \exists (\omega_n) \subseteq \mathcal{C}^\infty_c\Omega^j(M) \text{ s.t. } \omega_n \to \omega \text{ and } d\omega_n \text{ is } L^2\text{-Cauchy}\}.$$

[4]We follow "Hilbert Complexes" by Jochen Brüning and Matthias Lesch.

[5]An operator is closed if its graph is a closed subset of the product.

It is easy to see that each $(d, \mathscr{D}_{\min}(d)_j)$ is a closed operator and that they make up a Hilbert complex.

The L^2-cohomology of a Riemannian manifold only depends on g through the induced spaces of L^2-differential forms, so it is invariant under any change of metric that does not change the L^2 spaces.

Theorem 1.1. *If g and g' are Riemannian metrics on a manifold M and there is a constant C such that*

$$Cg \leq g' \leq C^{-1} g$$

then the maximal and minimal domains of d on (M, g) and (M, g') coincide. In particular so does the L^2-cohomology and reduced L^2-cohomology.

Two metrics satisfying the condition in the theorem are said to be **quasi-isometric**.

1.3. **Ideal boundary conditions.** If (M, g) is a compact Riemannian manifold with boundary then one can show[6] that the cohomology of the Hilbert complex of the minimal closed extension gives the cohomology of M relative to its boundary, and the cohomology of the Hilbert complex of the maximal extension gives the absolute cohomology of M,

$$\mathrm{H}^j((\mathscr{D}_{\min}(d)_\bullet, d)) \cong \mathrm{H}^j(M, \partial M), \quad \mathrm{H}^j((\mathscr{D}_{\max}(d)_\bullet, d)) = \mathrm{H}^j_{L^2}(M, g) \cong \mathrm{H}^j(M).$$

Because of this example, and following nomenclature of Cheeger for spaces with isolated conic singularities, any Hilbert complex with $H_j = L^2\Omega^j(M)$ and $(D_j, \mathcal{D}_j)$ an extension of $(\mathcal{C}_c^\infty \Omega^j(M), d)$ is known as a **choice of ideal boundary conditions.** If $\mathscr{D}_{\min}(d) = \mathscr{D}_{\max}(d)$ we say that M has **negligible boundary** (following Gafney) or that **the L^2-Stokes theorem holds** on M (following Cheeger).

The main example of spaces with negligible boundary are complete manifolds. To prove this we recall the following characterization of complete manifolds.

Lemma 1.2 (Gordon, de Rham, Borel)**.** *A smooth Riemannian manifold M is complete if and only if there is a smooth proper function $M \xrightarrow{\mu} [0, \infty)$ whose gradient is uniformly bounded.*

Proof. If M is complete and $p \in M$, note that the distance to p is a proper function, differentiable almost everywhere and with gradient of length one. We obtain μ by smoothing out $d(p, \cdot)$. Another approach with more machinery is to use the Nash embedding theorem to find an isometric embedding $J : M \to \mathbb{R}^N$. Then completeness of M is equivalent to properness of J. Define $F : \mathbb{R}^N \to \mathbb{R}$ by $F(\zeta) = \log(1 + |\zeta|^2)$ so that F is proper, smooth, and has gradient of length bounded by one, and note that the same is true for $\mu = F \circ J$.

Conversely, given such a function, let $\gamma : \mathcal{J} \subseteq \mathbb{R} \to M$ be a geodesic segment parametrized by arclength. If the length of γ is finite then, since $|\dot{\gamma}| = 1$, the variation of $\mu \circ \gamma$ on $\mathcal{J}$ is bounded. Because μ is proper this implies that the range of γ is contained in a compact subset of M and hence can be extended at both ends. Hence M is complete. $\square$

Theorem 1.3 (Gaffney)**.** *Let M be a complete Riemannian manifold, and consider a first order differential operator $D \in \mathrm{Diff}^1(M; E, F)$ satisfying $|\sigma(D)(\zeta, \xi)| \leq C(1 + |\xi|)$ uniformly on M, then*

[6] For example, it follows from an easier version of what we cover in the second lecture.

a) $\mathcal{D}_{\min}(D) = \mathcal{D}_{\max}(D)$,
b) $\mathcal{D}_{\min}(D^*D) = \mathcal{D}_{\max}(D^*D)$, *hence D^*D is essentially self-adjoint.*

Beyond the exterior derivative d, this also applies to the formal[7] adjoint of d, δ, any covariant derivative ∇, and more generally any 'Dirac-type' operator $\eth$. Applying part (b) to the operator $D = d + \delta$ shows that the Hodge Laplacian is essentially self-adjoint.

Proof.
[**a**)] We need to show that $\mathcal{D}_{\max}(D) \subseteq \mathcal{D}_{\min}(D)$, so we fix $u \in \mathcal{D}_{\max}(D) \subseteq H^1_{\text{loc}}(M)$ and we will find a sequence $(u_j) \subseteq \mathcal{D}(D)$ such that $u_j \to u$ and $Du_j \to Du$ in L^2. Fix $\chi \in \mathcal{C}^\infty(\mathbb{R}; [0,1])$ such that

$$\chi(t) = \begin{cases} 1 & \text{if } t \leq 0 \\ 0 & \text{if } t \geq 1 \end{cases}$$

and define $\chi_j(q) = \chi(\mu(q) - j)$, so that

$$\chi_j(q) = \begin{cases} 1 & \text{if } \mu(q) \leq j \\ 0 & \text{if } \mu(q) \geq j+1 \end{cases}$$

Notice that we can find $C > 0$ such that $|\chi_j| \leq C$ independently of j. Define $u_j = \chi_j u$ and note that $u_j \to u$ and

$$Du_j = D(\chi_j u) = \chi_j Du + [D, \chi_j]u = \chi_j Du + \sigma(D)(\cdot, d\chi_j)u$$

Clearly the first term on the right converges to Du. The second term converges pointwise to zero, since $[D, \chi_j]$ is supported in $\mu^{-1}([j, j+1])$, and its pointwise norm is bounded above by $C|u|_E$, hence by Lebesgue dominated convergence it converges to zero in $L^2(M, F)$.

[**b**)] Our strategy is to find an expression for $\mathcal{D}_\cdot(D^*D)$ in terms of the domains of D and D^* (both of which satisfy (a)). Let $u \in L^2(M, E)$ be an element of $\mathcal{D}_{\max}(D^*D)$ and denote $\chi_{j+1}u$ by u_{j+1}, note that

$$\langle \chi_j^2 u, D^*Du \rangle = \langle \chi_j^2 u_{j+1}, D^*D(u_{j+1}) \rangle$$

since χ_{j+1} is identically equal to one on the support of χ_j. For any $\phi \in \mathcal{C}_c^\infty(M)$ we know that ϕu is a compactly supported element of $H^2_{\text{loc}}(M; E)$ and hence is in the domain of both D^* and D. It follows that

$$\begin{aligned}
\langle \chi_j^2 u_{j+1}, D^*D(u_{j+1}) \rangle &= \langle D(\chi_j^2 u_{j+1}), D(u_{j+1}) \rangle \\
&= \langle \chi_j^2 D(u_{j+1}) + 2\chi_j \sigma(D)(d\chi_j)u_{j+1}, D(u_{j+1}) \rangle \\
&= \langle \chi_j D(u_{j+1}), \chi_j D(u_{j+1}) \rangle + \langle 2\chi_j \sigma(D)(d\chi_j)u_{j+1}, D(u_{j+1}) \rangle.
\end{aligned}$$

[7]By 'formal' adjoint of d we mean the differential operator δ satisfying

$$\langle d\omega, \phi \rangle = \langle \omega, \delta\phi \rangle$$

whenever ω and ϕ are smooth forms of compact support. The 'formal' moniker is used because the domain $\mathcal{C}_c^\infty \Omega^*(M)$ is not a closed domain in $L^2\Omega^*(M)$. In contrast, examples of adjoints include

$$(d, \mathscr{D}_{\max}(d))^* = (\delta, \mathscr{D}_{\min}(\delta)), \quad (d, \mathscr{D}_{\min}(d))^* = (\delta, \mathcal{D}_{\max}(\delta)),$$

where the minimal and maximal domains of δ are defined analogously to those of d.

However, it is easy to see that

$$\begin{aligned}\langle 2\chi_j\sigma(D)u_{j+1}, D(u_{j+1)}\rangle &\geq -\|\chi_j D(u_{j+1})\|\|2\sigma(D)u_{j+1}\| \\ &= \frac{1}{2}\left[(\|\chi_j D(u_{j+1})\| - \|2\sigma(D)u_{j+1}\|)^2 - \|\chi_j D(u_{j+1})\|^2 - \|2\sigma(D)u_{j+1}\|^2\right] \\ &\geq -\frac{1}{2}\|\chi_j D(u_{j+1})\|^2 - C'\|u_{j+1}\|^2,\end{aligned}$$

for some $C' > 0$, and we can conclude that

$$\begin{aligned}\frac{1}{2}\|\chi_j D(u_{j+1})\|^2 &\leq C'\|u_{j+1}\|^2 + \langle \chi_j^2 u_{j+1}, D^* D(u_{j+1})\rangle \\ &\leq C'\|u\|^2 + \langle u, D^* D u\rangle.\end{aligned}$$

It follows that $Du \in L^2(M,F)$ and $D^*(Du) \in L^2(M,E)$, i.e.,

$$\mathcal{D}_{\max}(D^*D) = \left\{u \in L^2(M,E) : u \in \mathcal{D}_{\max}(D) \text{ and } Du \in \mathcal{D}_{\max}(D^*)\right\}.$$

Now consider $u \in \mathcal{D}_{\min}(D^*D)$. Choose $u_j \in \mathcal{C}_c^\infty(M,E)$ such that both u_j and D^*Du_j converge in $L^2(M,E)$. Notice that

$$\|Du_j - Du_k\|^2 = \langle D^*D(u_j - u_k), u_j - u_k\rangle \to 0,$$

i.e., Du_j converges in $L^2(M,F)$ and hence $u \in \mathcal{D}_{\min}(D)$. Convergence of $D^*(Du_j)$ shows that $Du \in \mathcal{D}_{\min}(D^*)$ and hence

$$\mathcal{D}_{\min}(D^*D) = \left\{u \in L^2(M,E) : u \in \mathcal{D}_{\min}(D) \text{ and } Du \in \mathcal{D}_{\min}(D^*)\right\}.$$

By part (a), $\mathcal{D}_{\min}(D^*D) = \mathcal{D}_{\max}(D^*D)$.

□

If D is an operator covered by Gaffney's theorem, and $u, v \in \mathcal{D}(D^*D)$, then

$$\langle D^*Du, v\rangle = \langle Du, Dv\rangle.$$

In particular, for the Hodge Laplacian on k-forms on a complete manifold, this implies that

$$\langle \Delta\omega, \eta\rangle = \langle (d+\delta)\omega, (d+\delta)\omega\rangle = \langle d\omega, d\eta\rangle + \langle \delta\omega, \delta\eta\rangle.$$

In particular harmonic forms coincide with forms that are closed and co-closed, i.e.,

$$\mathrm{Ker}(\Delta) = \mathrm{Ker}(d) \cap \mathrm{Ker}(\delta).$$

Among incomplete manifolds there is often a topological condition required for L^2 Stokes theorem to hold. For a Riemannian space with an isolated conic singularity with link Z, Cheeger pointed out that the topological condition is that $\mathrm{H}^{\dim Z/2}(Z) = 0$, if $\dim Z$ is even. This is called the 'Witt condition'.

1.4. **Kodaira decomposition.** Every Hilbert complex $(\mathcal{D}_\bullet, D_\bullet)$ has a natural dual complex. Indeed, the adjoint operators

$$D_j^* : \mathcal{D}(D_j^*) \subseteq H_{j+1} \longrightarrow H_j$$

are closed and densely defined with domain

$$\mathcal{D}(D_j^*) = \{\omega \in H_{j+1} : H_j \ni \alpha \mapsto \langle D_j\alpha, \omega\rangle \in \mathbb{C} \text{ is bounded}\}$$

and satisfy

$$\mathcal{R}_j^* \subseteq \mathcal{D}_{j-1}^*, \quad D_{j-1}^* \circ D_j^* = 0.$$

The corresponding Hilbert complex is

$$0 \longleftarrow \mathcal{D}_{-1}^* \xleftarrow{D_0^*} \mathcal{D}_0^* \xleftarrow{D_1^*} \dots \xleftarrow{D_{N-1}^*} \mathcal{D}_{N-1}^* \longleftarrow 0.$$

The **Hodge cohomology groups** of a Hilbert complex $(\mathcal{D}_\bullet, D_\bullet)$ are the vector spaces

$$\mathcal{H}_j = \mathcal{H}_j((\mathcal{D}_\bullet, D_\bullet)) = \operatorname{Ker} D_j \cap \operatorname{Ker} D_{j-1}^*, \quad j \in \{0, \dots, N\}.$$

The Hodge cohomology groups of the dual complex are the same up to re-indexing

$$\mathcal{H}_j((\mathcal{D}_\bullet, D_\bullet)) = \mathcal{H}_{N-j}((\mathcal{D}_\bullet^*, D_\bullet^*))$$

Theorem 1.4. *(Weak Kodaira decomposition) Let $(\mathcal{D}_\bullet, D_\bullet)$ be a Hilbert complex. For each j there is an orthogonal decomposition*

$$H_j = \mathcal{H}_j \oplus \overline{\mathcal{R}}_{j-1} \oplus \overline{\mathcal{R}_j^*}$$

Proof. Since $(D_j, \mathcal{D}(D_j))$ is a closed operator we have

$$\operatorname{Ker} D_j = \overline{\operatorname{Ker} D_j}, \quad (\operatorname{Ker} D_j)^\perp = \overline{\operatorname{Im}(D_j^*, \mathcal{D}(D_j^*))} = \overline{\mathcal{R}_j^*}.$$

Thus we can write $H_j = \overline{\mathcal{R}_j^*} \oplus \operatorname{Ker} D_j$ and then further decompose

$$H_j = \overline{\mathcal{R}_j^*} \oplus \overline{\mathcal{R}}_{j-1} \oplus \operatorname{Ker} D_j \cap \operatorname{Ker} D_{j-1}^*.$$

□

In particular it follows that for any Hilbert complex

$$(\operatorname{Ker} D_j)/(\overline{\mathcal{R}_j}) \cong \operatorname{Ker} D_j \cap \operatorname{Ker} D_{j-1}^*.$$

For a Riemannian manifold, we define the L^2-Hodge cohomology groups to be

$$\mathcal{H}_{L^2}^j(M) = \{\omega \in L^2\Omega^j(M) : d\omega = 0 = \delta\omega\}.$$

Corollary 1.5. *Let (M, g) be a Riemannian manifold. There is a natural surjective map from the L^2-harmonic forms onto the reduced L^2-cohomology (or indeed, the reduced cohomology for any choice of ideal boundary conditions),*

$$\mathcal{H}_{L^2}^j(M) \cong \ker(d, \mathcal{D}_{\max}(d)) \cap \ker(\delta, \mathcal{D}_{\max}(\delta)) \longrightarrow \overline{\mathrm{H}}_{L^2}^j(M) \cong \ker(d, \mathcal{D}_{\max}(d)) \cap \ker(\delta, \mathcal{D}_{\min}(\delta)).$$

If M has negligible boundary then this map is an isomorphism.

We can also consider the space of L^2-harmonic forms. The Hodge Laplacian is the operator on j-forms given by $d\delta + \delta d$ and the L^2-harmonic forms are the elements of its distributional null space,

$$\operatorname{Ker}(\Delta_j, \mathcal{D}_{\max}(\Delta_j)) = \{\omega \in L^2\Omega^j(M) : (d\delta + \delta d)\omega = 0\}.$$

The L^2-harmonic forms contain the L^2-Hodge cohomology but are usually larger. For example, if M is equal to the unit interval, $M = (0, 1)$, then the zero$^{\text{th}}$ Hodge cohomology group consists of all constant functions while the harmonic functions are the polynomials of degree at most one.

1.5. Smooth subcomplex.

[8] Elliptic regularity shows that L^2-harmonic forms are smooth. In fact the L^2-cohomology can always be computed using a subcomplex of smooth forms. One way of showing this, by mollification, goes back to de Rham[9]. Here is an approach that works for all Hilbert complexes.

Theorem 1.6. *Let $(\mathcal{D}_\bullet, D_\bullet)$ be a Hilbert complex with corresponding domain for the Hodge Laplacian,*

$$\mathcal{D}(\Delta_j) = \{\omega \in H_j \cap \mathcal{D}_j \cap \mathcal{D}^*_{j-1} : d\omega \in \mathcal{D}^*_j \text{ and } \delta\omega \in \mathcal{D}_{j+1}\},$$

and define recursively

$$\mathcal{D}(\Delta_j^k) = \{\omega \in \mathcal{D}(\Delta_j) : \Delta_j \omega \in \mathcal{D}(\Delta_j^{k-1})\}, \quad \mathcal{D}(\Delta_j^\infty) = \bigcap \mathcal{D}(\Delta_j^k).$$

The Hilbert complex restricts to these subspaces to form a subcomplex and the inclusion map induces isomorphisms in cohomology.

Proof. Since $(\Delta_j, \mathcal{D}(\Delta_j))$ is a non-negative self-adjoint operator, the spectral theorem allows us to write it as an integral over $\mathbb{R}^+$ with respect to its spectral measure, $\Delta = \int_0^\infty \lambda \, dE_\lambda$. Given any $\omega \in \mathcal{D}(\Delta_j)$, we define

$$\omega_n = (\int_0^n dE_\lambda)\omega$$

and find that $\omega_n \in \mathcal{D}(\Delta_j^\infty)$, and $\omega_n \to \omega$ in the graph norm. Thus $\mathcal{D}(\Delta_j^\infty)$ is a 'core' for $\mathcal{D}(\Delta_j)$.

Both D_j and D_j^* commute with any power of the Hodge Laplacian, so they both preserve $\mathcal{D}(\Delta_j^\infty)$ and restriction yields a subcomplex.

Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ be equal to one near zero, and let

$$A_j = -D^*_{j-1}(\Delta_j + \mathrm{id})^{-1}\left(\mathrm{id} + \int_0^\infty \phi(t) e^{-t\Delta}\, dt\right).$$

One can show that

$$A_j : \mathcal{D}_j \longrightarrow \mathcal{D}_{j-1}, \quad A_j : \mathcal{D}(\Delta_j^\infty) \longrightarrow \mathcal{D}(\Delta_j^\infty)$$

and that the inclusion of $\mathcal{D}(\Delta_j^\infty)$ is equal to $\mathrm{id} + D_{j-1}A_j + A_{j+1}D_j$. Hence the subcomplex has the same homology as the full complex. □

2. Lecture Two: A de Rham theoretic approach

In this lecture we consider some examples, both complete and singular. The results in this section are due to Cheeger, Zucker, Hausel, Hunsicker, Mazzeo, and Bei, sometimes in collaboration.

Reference recommendation for this lecture:

- Jeff Cheeger, 'On the Hodge theory of Riemannian pseudomanifolds',

[8] In this section we follow Brüning-Lesch, 'Hilbert complexes'.

[9] De Rham, Variétés différentiables, pg. 72-82. See also the Appendix of 'On the Hodge Theory of Riemannian pseudomanifolds' by Jeff Cheeger.

- Francesco Bei, 'General perversities and L^2 de Rham and Hodge theorems for stratified pseudomanifolds',
- Boris Youssin, 'L^p cohomology of cones and horns",

Often properties of de Rham cohomology continue to hold in L^2 cohomology with extra functional analytic hypotheses. For example:

Proposition 2.1 (Künneth formula). [10] *Let $(H'_\bullet, D'_\bullet)$ and $(H''_\bullet, D''_\bullet)$ be two Hilbert complexes. Form the completed tensor product Hilbert complex $(H_\bullet, D_\bullet)$ where*

$$H_k = \bigoplus_{i+j=k} H'_i \widehat{\otimes} H''_j, \quad D_k = \bigoplus_{i+j=k} (D'_i \otimes \mathrm{id}_{H''_j} + (-1)^i \,\mathrm{id}_{H'_i} \otimes D''_j).$$

Assume that D''_* has closed range in all degrees, then

$$\mathrm{H}^k(H_\bullet, D_\bullet) = \bigoplus_{i+j=k} \mathrm{H}^i(H'_\bullet, D'_\bullet) \otimes \mathrm{H}^j(H''_\bullet, D''_\bullet).$$

In this lecture we will see another instance: the de Rham cohomology of $Z \times (0,1)$ is isomorphic to the de Rham cohomology of Z. We can sort of make this work in L^2 cohomology in some cases but the L^2-condition gives it a different character.

2.1. **Wedge and fibered cusp metrics.** Historically, L^2-cohomology was introduced independently by Cheeger and Zucker. Cheeger[11] was interested in stratified spaces and metrics with conic degeneration. For a simple example consider M, the interior of a manifold with boundary, and assume that the boundary of M participates in a fiber bundle,

$$Z - \partial M \xrightarrow{\phi} Y.$$

Fix a collar neighborhood $\mathscr{C}(\partial M) \cong [0,1) \times \partial M$ of ∂M in M, and denote the trivial extension of ϕ to $\mathscr{C}(\partial M)$ by the same symbol, ϕ. A **wedge metric** on M is one that in $\mathscr{C}(\partial M)$ takes the form

$$g_{\mathrm{w}} = dx^2 + x^2 g_Z + \phi^* g_Y,$$

where x is a boundary defining function, g_Z is a bundle metric for the vertical tangent bundle $T\partial M/Y$, and g_Y is a Riemannian metric on Y pulled-back after the choice of a connection for ϕ. Note that this metric has the effect of collapsing each of the fibers of ϕ to a point at $x = 0$. If Y is a point, this corresponds to an isolated conic singularity; if Z is a point, this corresponds to a smooth incomplete metric on a manifold with boundary.

Zucker[12] was interested in complete manifolds of finite volume with ends like hyperbolic cusps. A generalization, which includes the end of most locally symmetric spaces of $\mathbb{Q}$-rank one, is a metric on M as above that in a collar neighborhood of ∂M takes the form

$$g_d = \frac{dx^2}{x^2} + x^2 g_Z + \phi^* g_Y.$$

Note that this metric also has the effect of collapsing the fibers of ϕ to a point, but the denominator x^2 in the first term shows that the collapse happens 'at infinity'. If Y is a point, this corresponds to a complete manifold with finite volume and cusp ends, such as those

[10] See Brüning-Lesch, 'Hilbert complexes'.

[11] Jeff Cheeger, 'On the spectral geometry of spaces with cone-like singularities'

[12] Steven Zucker, 'Théorie de Hodge à coefficients dégénérescents'

occuring in hyperbolic geometry; if Z is a point, this corresponds to a complete manifold with cylindrical ends.

It is convenient to allow the Hilbert spaces to be weighted L^2 spaces of differential forms. For $\alpha \in \mathbb{R}$ consider

$$\mathscr{D}_{\max,\alpha}(d) = \{\omega \in x^\alpha L^2\Omega^j(M) : d\omega \in x^\alpha L^2\Omega^{j+1}(M)\},$$

the associated Hilbert complex

$$0 \longrightarrow \mathscr{D}_{\max,\alpha}(d)_0 \xrightarrow{d} \mathscr{D}_{\max,\alpha}(d)_1 \xrightarrow{d} \dots \xrightarrow{d} \mathscr{D}_{\max,\alpha}(d)_m \longrightarrow 0,$$

and the corresponding homology groups

$$\mathrm{H}^j(\mathscr{D}_{\max,\alpha}(d)_\bullet, d) = \mathrm{H}^j_{x^\alpha L^2}(M, g).$$

Part of the computation will be true for both metrics and more generally, so let us start by considering a metric on M that in a collar neighborhood has the form

$$g = g_{\beta,\gamma} = \frac{dx^2}{x^{2\beta}} + x^{2\gamma} g_Z + \phi^* g_Y,$$

we will restrict α, β, γ so that we are in the simplest setting[13], but this is not essential.

Remark 1. *As an exercise[14], consider the case of b-metrics: M is (the interior of) a manifold with boundary and the metric has the form $g_b = \frac{dx^2}{x^2} + g_{\partial M}$ near the boundary. Elements of $x^\alpha L^2\Omega^j(M)$ for $\alpha > 0$ have some degree of vanishing at ∂M and the $x^\alpha L^2$-cohomology is equal to the singular cohomology of M relative to its boundary,*

$$\mathrm{H}^j_{x^\alpha L^2}(M, g_b) = \mathrm{H}^j(M, \partial M) \text{ if } \alpha > 0.$$

Analogously, or rather 'dually', negative weights yield the absolute singular cohomology,

$$\mathrm{H}^j_{x^\alpha L^2}(M, g_b) = \mathrm{H}^j(M) \text{ if } \alpha < 0.$$

In the last remaining case, $\alpha = 0$, the image of d is not closed and so the L^2-cohomology is infinite dimensional. The reduced cohomology in this case is the image of relative into absolute,

$$\overline{\mathrm{H}}^j_{L^2}(M, g_b) = \mathrm{Image}\left(\mathrm{H}^j(M, \partial M) \longrightarrow \mathrm{H}^j(M)\right).$$

This lack of closed range is one of the issues we avoid below by placing restrictions on α, β, γ.

An important simplification in computing the weighted L^2 cohomology of $(M, g_{\beta,\gamma})$ is to identify it with the (hyper)cohomology of a sheaf complex. It turns out to be natural to work on the singular space associated to $\overline{M}$, $\widehat{M}$. This is the metric compactification of M with the wedge metric; it is obtained by collapsing the fibers of the boundary fibration. We define a complex of sheaves on $\overline{M}$ from the presheaf that assigns to each open set $\mathcal{U} \subseteq \overline{M}$ the vector space

$$\mathscr{D}_{\max,\alpha}(d; \mathcal{U}) = \{\omega \in \mathscr{D}_{\max,\alpha}(d) : \operatorname{supp} \omega \subseteq \mathcal{U} \cap M\}$$

and to each inclusion $j : \mathcal{V} \hookrightarrow \mathcal{U}$ the restriction map. We sheafify and denote the corresponding sheaf complex by $\mathbf{L}^2_{\max,\alpha}\mathbf{\Omega}^\bullet$.

[13]Specifically, we will assume that $\beta - 1 \leq 0 < \gamma$ and that $p = \frac{v}{2} + \frac{1-2\alpha-\beta}{2\gamma}$ is not an integer in $[0, m]$.

[14]To see this worked out, see Proposition 6.13 of The Atiyah-Patodi-Singer index theorem by Richard Melrose.

It is important that these sheaves are fine. To see this it suffices to construct a partition of unity using the functions

$$\mathcal{C}^\infty_\Phi(\overline{M}) = \{u \in \mathcal{C}^\infty(\overline{M}) : u\big|_{\partial M} \in \phi^*\mathcal{C}^\infty(Y)\}.$$

Indeed, if $u \in \mathcal{C}^\infty_\Phi(\overline{M})$ then $|du|$ is bounded and so multiplication by u preserves $\mathscr{D}_{\max,\alpha}(d)$.

It follows that the sheaf cohomology is determined by the stalk cohomology and in turn this is determined by the cohomology of 'distinguished neighborhoods'[15]. Each neighborhood has the form $\mathbb{B}^h \times (0,1) \times Z$ and since we may change the metric within its quasi-isometry class, we may assume that it is the product of the Euclidean metric on the ball and a metric $x^{-2\beta}dx^2 + x^{2\gamma}g_Z$, wherein g_Z is the same on each fiber of Z, over $\mathbb{B}^h$. A Künneth-type argument[16] shows that the cohomology of this neighborhood is isomorphic to the L^2-cohomology of $(0,1) \times Z$.

2.2. L^2-cohomology of a horn/cusp metric. [17]

Consider then $\mathscr{C}(Z) = (0,1) \times Z$ with the metric

$$\frac{dx^2}{x^{2\beta}} + x^{2\gamma}g_Z$$

for some fixed metric g_Z on Z. We want to compute the $x^\alpha L^2$-cohomology. As pointed out above we obtain the same cohomology by using differential forms that are smooth on $\mathscr{C}(Z)$, and since the metric is not singular at $x = 1$ a standard mollification argument shows that we can use forms that are smooth up to $x = 1$,

$$\mathscr{D}^\infty_{\max,\alpha}(d)_j = \{\omega \in \mathcal{C}^\infty((0,1] \times Z; \Lambda^j T^*\mathscr{C}(Z)) : \omega, d\omega \in x^\alpha L^2(\mathscr{C}(Z); \Lambda^* T^*\mathscr{C}(Z))\}.$$

A consequence of the construction in this section is that in some cases we can further restrict to forms that are smooth on all of $\overline{\mathscr{C}(Z)} = [0,1] \times Z$,

$$\mathscr{D}^{\overline{\infty}}_{\max,\alpha}(d)_j = \{\omega \in \mathcal{C}^\infty(\overline{\mathscr{C}(Z)}; \Lambda^j T^*\overline{\mathscr{C}(Z)}) : \omega, d\omega \in x^\alpha L^2(\mathscr{C}(Z); \Lambda^* T^*\mathscr{C}(Z))\},$$

in which case the cohomology is easy to identify.

Let us start by considering smooth differential forms without worrying about the L^2-condition. For any $a \in (0,1)$, consider the map[18]

$$T_a : \mathcal{C}^\infty(\mathscr{C}(Z); \Lambda^j T^*\mathscr{C}(Z)) \longrightarrow \mathcal{C}^\infty(\mathscr{C}(Z); \Lambda^{j-1} T^*\mathscr{C}(Z)), \quad (T_a\omega)(x,\zeta) = \int_a^x \mathfrak{i}_{\partial_x}\omega(t,\zeta)\, dt,$$

i.e., in terms of the decomposition of ω as $\omega = \omega_t + dx \wedge \omega_n$, T_a is integration of ω_n. Note that

$$(dT_a\omega + T_a d\omega)(x,\zeta) = \omega(x,\zeta) - \pi_a^*\omega_t(\zeta), \tag{2.1}$$

where $\pi_a^*\omega_t(\zeta) = \omega_t(a,\zeta)$, and hence the complex of smooth differential forms is quasi-isomorphic to the complex of smooth forms on Z pulled-back from $\{a\} \times Z$, and so are in

[15] These neighborhoods arise naturally when considering the stratified space $\widehat{M}$. Just as points on a smooth manifold are required to have a neighborhood that looks like an open ball in $\mathbb{R}^m$, points on stratified spaces have neighborhoods that look like an open ball in $\mathbb{R}^h$ (a coordinate chart on Y, the singular stratum), times the cone over another space, so $\mathbb{B}^h \times C(Z)$. The intersection with the regular part is thus $\mathbb{B}^h \times (0,1) \times Z$.

[16] Proposition 2.1 works with $(H''_\bullet, D_\bullet)$ the Hilbert complex for the L^2-cohomology of $\mathbb{B}^h$. We know that the closed range condition holds because the L^2-cohomology is finite dimensional.

[17] Our approach follows Cheeger, Youssin, Bei.

[18] We use T for 'transgression'.

particular smooth on $\overline{\mathscr{C}(Z)}$.

When will these maps be bounded in $x^\alpha L^2$?

If $\omega \in \mathscr{D}^\infty_{\max,\alpha}(d)_j$, then the pointwise norm of ω is given by

$$|\omega|^2_{g(x,\zeta)} = x^{-2j\gamma}|\omega_t|^2_{g(1,\zeta)} + x^{2\beta}x^{-2(j-1)\gamma}|\omega_n|^2_{g(1,\zeta)}$$

and we get the $x^\alpha L^2$ norm squared by integrating against $x^{-2\alpha}$ times the volume form

$$(2.2) \quad \int_{\mathscr{C}(Z)} |\omega|^2_{g(x,p)} \; x^{v\gamma}x^{-\beta}x^{-2\alpha}dx \, \mathrm{dvol}_Z = \int_0^1 \left(x^{\gamma(v-2j)-\beta-2\alpha}\|\omega_t(x)\|_1^2 + x^{\gamma(v-2(j-1))+\beta-2\alpha}\|\omega_n(x)\|_1^2\right) \, dx.$$

The vertical degree of $T_a\omega$ coincides with that of ω_n, so is equal to $j-1$, and its $x^\alpha L^2$-norm satisfies

$$\|T_a\omega\|^2_{x^\alpha L^2} = \int_0^1 x^{\gamma(v-2(j-1))-\beta-2\alpha}\|T_a\omega\|_1^2 \, dx \leq \int_0^1 x^{\gamma(v-2(j-1))-\beta-2\alpha}\left|\int_a^x \|\omega_n(t)\|_1 \, dt\right|^2 \, dx$$

Lemma 2.2. [19] *Let $f(r) \in L^2((0,1); x^\mu \, dx)$.*
i) If $\mu < -1$ then $x \mapsto \displaystyle\int_0^x f(t) \, dt$ is in $L^2((0,1); x^\mu \, dx)$.
ii) If $\mu \geq -1$ then $x \mapsto \displaystyle\int_1^x f(t) \, dt$ is in $L^2((0,1); x^\mu \, dx)$.
In either case the map $f \mapsto \int_a^x f(t) \, dt$ is a bounded map.

Proof. i) For any positive function λ, we have

$$\left|\int_0^x f(t) \, dt\right|^2 \leq \left(\int_0^x \lambda(s) \, ds\right)\left(\int_0^x \frac{f(t)^2}{\lambda(t)} \, dt\right),$$

hence

$$\int_0^1 \left|\int_0^x f(t) \, dt\right|^2 x^\mu \, dx \leq \int_0^1 \left(\int_0^x \lambda(s) \, ds\right) x^\mu \left(\int_0^x \frac{f(t)^2}{\lambda(t)} \, dt\right) \, dx$$
$$= \int_0^1 \frac{f(t)^2}{\lambda(t)} \int_t^1 \left(\int_0^x \lambda(s) \, ds\right) x^\mu \, dxdt$$

and we see that it suffices to find a positive function λ such that

$$\frac{1}{t^\mu \lambda(t)} \int_t^1 \left(\int_0^x \lambda(s) \, ds\right) x^\mu \, dx \text{ is bounded.}$$

Since $\mu < -1$ we can find $\varepsilon \in (2, 1-\mu)$ and consider $\lambda(t) = t^{-\mu-\varepsilon}$ which yields

$$t^\varepsilon \int_t^1 \left(\int_0^x s^{-\mu-\varepsilon} \, ds\right) x^\mu \, dx = t^\varepsilon \int_t^1 \frac{1}{1-\mu-\varepsilon} x^{-\varepsilon+1} \, dx = \frac{1}{(1-\mu-\varepsilon)(2-\varepsilon)}(t^\varepsilon - t^2)$$

[19] Following Proposition 3.39 of L_2 *Cohomology of Warped Products and Arithmetic Groups* by Steven Zucker.

which is bounded.
ii) Similar reasoning shows that is suffices to find a positive function λ such that

$$\frac{1}{t^\mu \lambda(t)} \int_0^t \left(\int_1^x \lambda(s)\, ds \right) x^\mu \, dx \text{ is bounded.}$$

For $\mu > -1$ we can take $\lambda(t) = t^{-\mu-\varepsilon}$ with $\varepsilon \in (1-\mu, 2)$, and for $\mu = -1$ we can take $\lambda(t) = t^{-1} \log^{-\varepsilon} t$ for any $\varepsilon \in (1,2)$. □

Next let us combine this with (2.2). Let

$$p = p(\alpha, \beta, \gamma) = \frac{v}{2} + \frac{1 - 2\alpha - \beta}{2\gamma}. \tag{2.3}$$

Lemma 2.3.
a) If $\omega \in \mathscr{D}^\infty_{\max,\alpha}(d)_j$ then $\pi_1^*\omega$, the pull-back of its restriction to $\{x = 1\}$, will be in $x^\alpha L^2$ if and only if

$$\gamma(v - 2j) - \beta - 2\alpha > -1 \iff k < p.$$

b) If $\omega \in \mathscr{D}^\infty_{\max,\alpha}(d)_j$, so that ω_n has degree $j-1$, then

$$\begin{cases} j - 1 \leq p \implies T_1\omega \in L^2\Omega^*(M) \\ j - 1 > p \implies T_0\omega \in L^2\Omega^*(M) \end{cases}$$

We are now in good shape to apply the formula (2.1) to forms with vertical degree less than p. For forms of larger vertical degree we need to deal with the fact that the pull-back is not an L^2 differential form. We do this by approximating T_0 as follows[20].

Let $\mathcal{K}_\varepsilon : (0,1] \longrightarrow (0, \mathcal{K}_\varepsilon(1)]$, $\varepsilon > 0$, be a one-parameter family of diffeomorphisms such that $0 < \mathcal{K}_\varepsilon(x) \leq x$ and $\mathcal{K}_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Let

$$\widetilde{T}_\varepsilon : \mathcal{C}^\infty(\mathscr{C}(Z); \Lambda^k T^*\mathscr{C}(Z)) \longrightarrow \mathcal{C}^\infty(\mathscr{C}(Z); \Lambda^{k-1} T^*\mathscr{C}(Z)), \quad (\widetilde{T}_\varepsilon \omega)(x, \zeta) = \int_{\mathcal{K}_\varepsilon(x)}^x \mathfrak{i}_{\partial_x} \omega(t, \zeta)\, dt.$$

Analyzing this as above yields the following:

Lemma 2.4. *If $\omega \in \mathscr{D}^\infty_{\max,\alpha}(d)$ then, as smooth forms on $\mathscr{C}(Z)$,*

$$(d\widetilde{T}_\varepsilon \omega + \widetilde{T}_\varepsilon d\omega)(x, p) = \omega - \mathcal{K}_\varepsilon^* \omega,$$

where $\mathcal{K}_\varepsilon^*\omega$ denotes the pull-back by the map induced by $\mathcal{K}_\varepsilon$ on $\mathscr{C}(\partial M)$.
If moreover $\omega_n = \mathfrak{i}_{\partial_x}\omega$ is an L^2 differential form of degree greater than p, then

$$\|\widetilde{T}_\varepsilon \omega - T_0 \omega\|_{L^2} \to 0 \text{ as } \varepsilon \to 0.$$

Choosing $\mathcal{K}_\varepsilon$ carefully lets us easily analyze the remainder term $\mathcal{K}_\varepsilon^*\omega$.

Lemma 2.5. *Let $\omega \in \mathscr{D}^\infty_{\max,\alpha}(d)_j$ with*

$$j > p + 1, \quad j \geq \frac{v+1}{2} - \frac{\alpha}{\gamma},$$

then, as $\varepsilon \to 0$,

$$\mathcal{K}_\varepsilon^* \omega \to 0, \quad \widetilde{T}_\varepsilon \omega \to T_0 \omega \text{ in } x^\alpha L^2.$$

It follows that

$$(dT_0 + T_0 d)\omega = \omega.$$

[20] We argue as in §5 of *L^p cohomology of cones and horns* by Boris Youssin.

Proof. By assumption $j > p$ is such that

$$u(x) = \int_1^x t^{\gamma(v-2k_t)-\beta-2\alpha}\, dt$$

diverges as $x \to 0$, so u takes $(0,1)$ to $(-\infty, 0)$.

Pulling ω back to $(-\infty, 0)$ by the inverse of u yields

$$\omega_t(x(u), \zeta) + \frac{du}{(x(u))^{\gamma(v-2j)-\beta-2\alpha}} \wedge \omega_n(x(u), \zeta)$$

which we denote $\omega_t + du \wedge \widetilde{\omega}_n$. The $x^\alpha L^2$ norm in terms of du is given by

$$\begin{aligned}\|\omega\|^2_{x^\alpha L^2} &= \int_0^1 \left(x^{\gamma(v-2j)-\beta-2\alpha}\|\omega_t(x)\|_1^2 + x^{\gamma(v-2(j-1))+\beta-2\alpha}\|\omega_n(x)\|_1^2\right)\, dx \\ &\xrightarrow{du = x^{\gamma(v-2j)-\beta-2\alpha}\, dx} \int_{-\infty}^0 \left(\|\omega_t(x(u))\|_1^2 + x(u)^{2\gamma+2\beta}\|\omega_n(x(u))\|_1^2\right)\, du \\ &= \int_{-\infty}^0 \left(\|\omega_t(x(u))\|_1^2 + x(u)^{2\gamma(v-(2j-1))-4\alpha}\|\widetilde{\omega}_n(x(u))\|_1^2\right)\, du.\end{aligned}$$

For $\varepsilon \in (0,1)$, define $K_\varepsilon : (-\infty, 0] \longrightarrow (-\infty, -\frac{1}{\varepsilon}]$ by $K_\varepsilon(u) = u - \frac{1}{\varepsilon}$. By assumption $2\gamma(v-(2j-1)) - 4\alpha \leq 0$ so $\mathcal{K}_\varepsilon^*(x(u)^{2\gamma(v-(2j-1))-4\alpha}) \geq x(u)^{2\gamma(v-(2j-1))-4\alpha}$ and the $x^\alpha L^2$ norm of $\mathcal{K}_\varepsilon^*\omega$ satisfies

$$\begin{aligned}\|K_\varepsilon^*\omega\|^2_{x^\alpha L^2} &= \int_{-\infty}^0 \left(\|\mathcal{K}_\varepsilon^*\omega_t(x(u))\|_1^2 + x(u)^{2\gamma(v-(2j-1))-4\alpha}\|\mathcal{K}_\varepsilon^*\widetilde{\omega}_n(x(u))\|_1^2\right)\, du \\ &\leq \int_{-\infty}^0 \left(\|\mathcal{K}_\varepsilon^*\omega_t(x(u))\|_1^2 + \mathcal{K}_\varepsilon^* x(u)^{2\gamma(v-(2j-1))-4\alpha}\|\mathcal{K}_\varepsilon^*\widetilde{\omega}_n(x(u))\|_1^2\right)\, du \\ &= \int_{-\infty}^{-1/\varepsilon} \left(\|\omega_t(x(u))\|_1^2 + x(u)^{2\gamma(v-(2j-1))-4\alpha}\|\widetilde{\omega}_n(x(u))\|_1^2\right)\, du.\end{aligned}$$

Comparing this to the $x^\alpha L^2$-norm of ω we see that

$$\|K_\varepsilon^*\omega\|^2_{x^\alpha L^2} \to 0 \text{ as } \varepsilon \to 0.$$

On $\{x > \delta\}$ for any $\delta > 0$ we have $(T_\varepsilon d + dT_\varepsilon)(\omega) = \omega - \chi K_\varepsilon^*\omega$ and we have shown that

$$K_\varepsilon^*\omega \to \omega, \quad T_\varepsilon\omega \to T_0\omega.$$

The form $(d\omega)_n$ also has degree greater than p and hence

$$T_\varepsilon d\omega \to T_0 d\omega.$$

It follows that

$$dT_\varepsilon\omega = \omega - \chi K_\varepsilon^*\omega - T_\varepsilon d\omega \to \omega - T_0 d\omega,$$

so that $T_0\omega \in \mathscr{D}_{\max}(d)$ with $dT_0\omega = \omega - T_0 d\omega$. □

Finally, let us consider what happens when $j - 1 \leq p < j$.

Lemma 2.6. *Let $j \in \mathbb{N}_0$ be such that $p \in [j-1, j)$ and assume that*

$$\operatorname{Im}(d : \mathscr{D}_{\max}(d_Z)_{j-1} \longrightarrow \mathscr{D}_{\max}(d_Z)_j) \text{ is closed.}$$

If $\omega \in \mathscr{D}^\infty_{\max,\alpha}(d)_j$ then there is a form $\phi \in \mathscr{D}^\infty_{\max}(d_Z)_{j-1}$ such that $d(\pi_1^*\phi + T_1\omega) + T_0 d\omega = \omega$

Proof. On $\{x > \delta\}$ for any $\delta > 0$, since $d\omega = d\omega_t + dx \wedge (\partial_x \omega_t - d_Z \omega_n)$, we have

$$\widetilde{T}_\varepsilon d\omega(x,\zeta) = \omega_t(x,\zeta) - \omega_t(K_\varepsilon(x),\zeta) - \int_{K_\varepsilon(1)}^x d_Z\omega_n(t,\zeta)\, dt.$$

The argument in the proof of Lemma 2.5 shows that $K_\varepsilon^* \omega_t \to 0$ in $x^\alpha L^2$ as $\varepsilon \to 0$. (Note that we do not need $k \geq \frac{v+1}{2} - \frac{\alpha}{\gamma}$ to conclude that the tangential part of $K_\varepsilon^*\omega \to 0$.) Since the normal part of $d\omega$ has degree greater than p, $\widetilde{T}_\varepsilon d\omega \longrightarrow T_0 d\omega$. Thus we can take the limit as $\varepsilon \to 0$ in the equality above to get

$$\lim_{\varepsilon\to 0} \int_{K_\varepsilon(1)}^x d_Z\omega_n(t,\zeta)\, dt = \omega_t(x,\zeta) - T_0 d\omega(x,\zeta).$$

Similarly,

$$dT_1\omega(x,\zeta) = dx \wedge \omega_n(x,\zeta) + \int_1^x d_Z\omega_n(t,\zeta)\, dt$$

and hence

$$(dT_1\omega + T_0 d\omega)(x,\zeta) = \omega(x,\zeta) - \lim_{\varepsilon\to 0}\int_{K_\varepsilon(1)}^1 d_Z\omega_n(t,\zeta)\, dt.$$

By our assumption this limit is equal to $d\pi^*\phi$ for some $\phi \in \mathscr{D}_{\max}(d_Z)_{j-1}$ □

We now add an assumption relating β and γ and put these results together to obtain the L^2-cohomology of $\mathscr{C}(Z)$.

Proposition 2.7. *Let $\mathscr{C}(Z) = (0,1) \times Z$ with the metric*

$$\frac{dx^2}{x^{2\beta}} + x^{2\gamma} g_Z$$

and assume that $\beta - 1 \leq 0 < \gamma$. Let $p = \frac{v}{2} + \frac{1-2\alpha-\beta}{2\gamma}$ and assume that $p \notin \{0, 1, \dots, m\}$. The $x^\alpha L^2$ cohomology of $\mathscr{C}(Z)$ is equal to

$$\mathrm{H}^j_{x^\alpha L^2}(\mathscr{C}(Z)) = \begin{cases} \mathrm{H}^j_{L^2}(Z) & \text{if } j < p \\ 0 & \text{if } j > p \end{cases}$$

Proof. If $\omega \in \mathscr{D}^\infty_{\max,\alpha}(d)_j$ with $j < p$ then, as smooth forms, we have the equality

$$(dT_1 + T_1 d)(\omega) = \omega - \pi_1^*\omega.$$

Lemma 2.3 shows that $T_1 d\omega$ and $\pi_1^*\omega$ are smooth forms in $x^\alpha L^2\Omega^j(\mathscr{C})$, and hence so is $dT_1\omega$. This shows that $\mathrm{H}^j_{x^\alpha L^2}(\mathscr{C}(Z)) \cong \mathrm{H}^j_{L^2}(Z)$.

If $\omega \in \mathscr{D}^\infty_{\max,\alpha}(d)_j$ with $j-1 < p < j$ then Lemma 2.6 applies and shows that

$$\omega = d(\pi_1^*\phi + T_1\omega) + T_0 d\omega.$$

In particular, $\mathrm{H}^j_{x^\alpha L^2}(\mathscr{C}(Z)) = 0$.

If $\omega \in \mathscr{D}^\infty_{\max,\alpha}(d)_j$ with $j > p+1$ then, since the assumption $\beta - 1 \leq 0 < \gamma$ guarantees that $p + 1 > \frac{v+1}{2} - \frac{\alpha}{\gamma}$, we can apply Lemma 2.5 and see that

$$\omega = (dT_0 + T_0 d)\omega.$$

In particular, $\mathrm{H}^j_{x^\alpha L^2}(\mathscr{C}(Z)) = 0$. □

2.3. **Intersection cohomology.** Let M be the interior of a manifold with fibered boundary $Z - \partial M \xrightarrow{\phi} Y$. At the boundary we have the vertical tangent bundle

$$T\partial M/Y = \ker D\phi$$

in terms of which we define the Cartan filtration:

$$F^k\Omega^*(\partial M) = \{\omega \in \Omega^*(\partial M) : \text{ for all } V_1, \dots, V_{k+1} \in \mathcal{C}^\infty(\partial M; T\partial M/Y), \mathrm{i}_{V_1} \cdots \mathrm{i}_{V_{k+1}}\omega = 0\}.$$

Thus $F^k\Omega^*(\partial M)$ consists of those forms on ∂M whose vertical degree is at most k.

Fix $\overline{p} \in \mathbb{Z}$ and define

$$\Omega^*_{\overline{p}}(\overline{M}) = \{\omega \in \Omega^*(\overline{M}) : i^*_{\partial M}\omega, i^*_{\partial M}d\omega \in F^{\overline{p}}\Omega^*(\partial M)\}.$$

This subcomplex of the de Rham complex was introduced by Brylinski-Goresky-MacPherson[21] and its elements are known as **intersection forms** with $\overline{p}$ the **perversity function**. The homology of this complex is known as the **intersection cohomology** of M with perversity $\overline{p}$ and is denoted

$$\mathrm{IH}^j_{\overline{p}}(M).$$

We can make a sheaf out of the intersection form complex much like we did for the L^2-cohomology. The 'local computation' in this context is much easier and yields

$$\mathrm{H}^j(\Omega_{\overline{p}}(\mathbb{B}^h \times [0,1) \times Z), d) = \begin{cases} \mathrm{H}^j(Z) & \text{if } j \leq \overline{p} \\ 0 & \text{if } j > \overline{p} \end{cases}$$

Comparing this with our local computation above we have identified the $x^\alpha L^2$ cohomology.

Theorem 2.8. *Endow M with a Riemannian metric that near the boundary takes the form*

$$g = \frac{dx^2}{x^{2\beta}} + x^{2\gamma}g_Z + \phi^*g_Y$$

satisfying $\beta - 1 \leq 0 < \gamma$. Let $\alpha \in \mathbb{R}$ be such that $p = \frac{v}{2} + \frac{1-2\alpha-\beta}{2\gamma} \notin \{0, \dots, m\}$ and let $\overline{p}$ be the largest integer smaller than p, then

$$\mathrm{H}^j_{x^\alpha L^2}(M, g) \cong \mathrm{IH}^j_{\overline{p}}(M).$$

3. Lecture Three: A Hodge theoretic approach

Suggested references for this lecture include:

- Tamas Hausel, Eugénie Hunsicker, and Rafe Mazzeo, 'Hodge cohomology of gravitational instantons',
- Richard Melrose, The Atiyah-Patodi-Singer index theorem,
- Pierre Albin, 'On the Hodge theory of stratified spaces'
- Jeff Cheeger, 'On the spectral geometry of spaces with cone-like singularities',
- Eugénie Hunsicker and Frédéric Rochon, 'Weighted Hodge cohomology of iterated fibred cusp metrics',
- Francesco Bei, 'General perversities and L^2 de Rham and Hodge theorems for stratified pseudomanifolds'

[21]Introduced in 'Equivariant intersection cohomology' by Jean-Luc Brylinski.

3.1. **Stratified spaces.** The first thing to point out is that the natural setting for the computation we did last lecture is stratified spaces. Examples of stratified spaces include algebraic varieties, orbit spaces of group actions, and some natural compactifications of moduli spaces and locally symmetric spaces[22].

A stratified space[23], $\widehat{X}$, is a topological space that is usually singular but is equal to a union of smooth manifolds. We will assume that the space if compact for simplicity.

Every point on a stratified space has a '**depth**' in $\mathbb{N}_0$. A point has depth zero if it has a neighborhood homeomorphic (diffeomorphic) to a ball in $\mathbb{R}^m$. If every point has depth zero, then the stratified space is actually a smooth manifold. A point has depth one if it does not have depth zero but instead small neighborhoods are homeomorphic to a Euclidean ball times the cone over a smooth manifold, $\mathbb{B}^h \times C(Z)$. The space Z is called the **link** of $\widehat{X}$ at that point. We define the **depth of a space** to be the maximal depth among its points. The space $\widehat{M}$ from the previous lecture has depth one. In general, a point has depth k if it does not have depth $k-1$ and small neighborhoods are homeomorphic to the product of a Eulidean ball times the cone over a stratified space of depth $k-1$.

That is the local picture of a stratified space. Globally, $\widehat{X}$ is a union of smooth manifolds, called the **strata** of $\widehat{X}$, and we need to describe how they fit together. The points of depth zero in $\widehat{X}$ make up the **regular part of** $\widehat{X}$, which we will denote X or $\widehat{X}^{\mathrm{reg}}$, this is a smooth manifold (usually open) and is assumed to be dense in $\widehat{X}$. Any other stratum[24] Y is referred to as a **singular stratum**, and comes equipped with a neighborhood in $\widehat{X}$, $\mathcal{T}_Y$, participating in a fiber bundle,

$$C(\widehat{Z}) - \mathcal{T}_Y \xrightarrow{\phi_Y} Y,$$

where the fiber is the cone over a stratified space $\widehat{Z}$, known as the **link of** Y **in** $\widehat{X}$. ($\widehat{Z}$ is a simpler stratified space than $\widehat{X}$, in that it has smaller depth.)

Remark 2. *As an exercise, show that 'the link of a link is a link'. That is, if $p \in \widehat{X}$ is a singular point on a stratified space and $\widehat{Z}$ is the link of p in $\widehat{X}$, if $q \in \widehat{Z}$ is a singular point and $\widehat{W}$ is the link of q in $\widehat{Z}$, then there is a point $r \in \widehat{X}$ whose link in $\widehat{X}$ is $\widehat{W}$.*

3.2. **The resolution of a stratified space.** There is an algorithm, going back at least as far as one of Thom's papers on stratified spaces[25], that resolves a stratified space into a manifold with corners. In the case of depth one, this corresponds to replacing $\widehat{M}$ from the previous lecture with $\overline{M}$, a manifold with fibered boundary.

Our preferred way of thinking about the resolution is due to Richard Melrose[26] and consists of a manifold with corners with an **iterated fibration structure**.

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[22]See, e.g., §2.1 of 'On the Hodge theory of stratified spaces' by P.A. .

[23]There are many notions of 'stratified space'. A fun place to read about the various types is 'Quelques notions d'espaces stratifiés' by Benoît Kloeckner. We work with 'smoothly stratified spaces', also known as Thom-Mather stratified spaces.

[24]We assume that $\dim Y < \dim X - 1$. Spaces satisfying this condition are known as **pseudomanifolds**.

[25]Specifically, 'Ensembles et Morphismes Stratifiés' by René Thom.

[26]This is worked out in the equivariant setting in 'Resolution of smooth group actions' by P.A. and Richard Melrose, and in the stratified setting in 'The signature package on Witt spaces' by P.A., Eric Leichtnam, Rafe Mazzeo, and Paolo Piazza. A discussion can be found in §6 of 'On the Hodge theory of stratified spaces' by P.A. .

An iterated fibration structure on $\widetilde{X}$, a manifold with corners, consists of fiber bundles,

$$\widetilde{Z} - \mathfrak{B}_Y \xrightarrow{\phi_Y} \widetilde{Y},$$

where the total space $\mathfrak{B}_Y$ is a union of pairwise disjoint boundary hypersurfaces (a '**collective boundary hypersurface**'), and the base and fiber are both manifolds with corners. Every boundary hypersurface of $\widetilde{X}$ occurs in exactly one of these collective boundary hypersurfaces and the fiber bundles are required to satisfy a compatibility condition:
If $\mathfrak{B}_Y \cap \mathfrak{B}_{Y'} \neq \emptyset$, then

(1) $\dim \widetilde{Y} \neq \dim \widetilde{Y}'$ and
(2) if $\dim \widetilde{Y} < \dim \widetilde{Y}'$ then there is a collective boundary hypersurface $\mathfrak{B}_{YY'}$ of $\widetilde{Y}'$ and a fiber bundle map $\mathfrak{B}_{YY'} \xrightarrow{\phi_{YY'}} \widetilde{Y}$ such that the following diagram commutes,

$$\begin{array}{ccc} \mathfrak{B}_Y \cap \mathfrak{B}_{Y'} & \xrightarrow{\phi_{Y'}} & \mathfrak{B}_{YY'} \\ & {\scriptstyle \phi_Y} \searrow \quad \swarrow {\scriptstyle \phi_{YY'}} & \\ & \widetilde{Y} & \end{array}$$

The bases of the fiber bundles,

$$\mathcal{S}(\widetilde{X}) = \{\widetilde{Y}\}$$

are in one-to-one correspondence with the singular strata of the stratified space $\widehat{X}$. In fact the closure of a singular stratum Y in $\widehat{X}$ is a stratified space $\widehat{Y}$, and, as the notation is meant to suggest, each $\widetilde{Y}$ is the resolution of some $\widehat{Y}$. Similarly, the fiber, $\widetilde{Z}$, of the boundary fiber bundle $\mathfrak{B}_Y \xrightarrow{\phi_Y} \widetilde{Y}$, is the resolution of the link of Y in $\widehat{X}$. This highly iterative structure lends itself naturally to inductive arguments.

The space $\widetilde{X}$ is obtained from $\widehat{X}$ by iteratively blowing-up a stratum of maximal depth. The singular space $\widehat{X}$ can be recovered from $\widetilde{X}$ by collapsing the fibers of the boundary fibrations in the appropriate order.

3.3. **Intersection cohomology.** As we did before, we define the intersection cohomology directly on the resolved space. Fix a function $\overline{p} : \mathcal{S}(\widetilde{X}) \longrightarrow \mathbb{Z}$, known as a '**perversity**'[27], and define

$$\Omega^*_{\overline{p}}(\widetilde{X}) = \{\omega \in \Omega^*(\widetilde{X}) : \text{ for all } \widetilde{Y} \in \mathcal{S}(\widetilde{X}),\ i^*_{\mathfrak{B}_Y}\omega, i^*_{\mathfrak{B}_Y} d\omega \in F^{\overline{p}(\widetilde{Y})}\Omega^*(\mathfrak{B}_Y)\}.$$

This clearly forms a complex and its cohomology is known as the intersection cohomology of $\widetilde{X}$ (or $\widehat{X}$) with perversity $\overline{p}$, denoted

$$\mathrm{IH}^*_{\overline{p}}(\widetilde{X}).$$

The results that we proved in the second lecture, and the proof, extend to stratified spaces. Cheeger showed that for a wedge metric the L^2 cohomology is an intersection cohomology, and Bei proved this for horn metrics. Hunsicker and Rochon proved this for d-metrics[28].

[27]A perversity is 'classical' if each $\overline{p}(\widetilde{Y})$ only depends on $\mathrm{cod}(\widetilde{Y}) = \dim \widetilde{X} - \dim \widetilde{Y}$, the codimension of $\widetilde{Y}$, and both $\overline{p}(c)$ and $\overline{q}(c) = 2 - c - \overline{p}(c)$ are non-negative and non-decreasing. For classical perversities, Goresky and MacPherson showed that intersection homology is a topological invariant.

[28]See the references at the beginning of the lecture

3.4. **The need for a Hodge theoretic approach.** There are (at least) two reasons why it is useful to take a Hodge theoretic approach to go beyond what we have established so far.

The first reason we want to emphasize is *Poincaré duality for L^2-cohomology of incomplete metrics.*

Recall that on a closed orientable manifold, M, the intersection pairing of singular cohomology is realized at the level of differential forms by

$$\Omega^*(M) \times \Omega^*(M) \xrightarrow{(\cdot,\cdot)} \mathbb{R}, \quad (u, v) = \int_M u \wedge v.$$

This pairing is non-degenerate, since $\int_M u \wedge *u = \|u\|^2_{L^2}$. Stokes' theorem guarantees that the pairing of an exact form and a closed form vanishes, i.e.,

$$dv = 0 \implies \int_M dw \wedge v = \int_M d(w \wedge v) = 0.$$

Thus this pairing descends to a non-degenerate pairing in cohomology. The signature of this pairing is known as the signature of the manifold and is an important invariant in manifold topology.

Notice that this fails when (M, g) is the restriction to the interior of a smooth Riemannian manifold with boundary. In this case, if $u, v \in \mathcal{D}_{\max}(d)$ are smooth up to the boundary, with $u = dw$ and v closed, then we have

$$\int_M dw \wedge v = \int_M d(w \wedge v) = \int_{\partial M} w \wedge v.$$

Thus the pairing does not descend to a non-degenerate pairing in cohomology. Of course, it does descend to a pairing between the maximal cohomology and the minimal cohomology,

$$\begin{array}{ccc} \mathrm{H}_*((\mathscr{D}_{\max}(d), d)) \times \mathrm{H}_*((\mathscr{D}_{\min}(d), d)) & \longrightarrow & \mathbb{R} \\ ([u], [v]) & \longmapsto & \int_M u \wedge v \end{array},$$

because by approximating v with compactly supported forms we see that there is no boundary term. This is the usual de Rham form of Poincaré-Lefschetz duality.

Now let (M, g) be an arbitrary Riemannian manifold. The intersection pairing is non-degenerate on $L^2\Omega^*(M, g) \times L^2\Omega^*(M, g)$. Every choice of closed domain $\mathscr{D}_{\min}(d) \subseteq \mathscr{D}(d) \subseteq \mathscr{D}_{\max}(d)$ determines a second domain $\mathscr{D}(d)'$ by

$$\mathscr{D}(d)' = \{v \in \mathscr{D}_{\max}(d) : \int_M d(u \wedge v) = 0 \text{ for all } u \in \mathscr{D}(d)\}.$$

This is a dense subset of $L^2\Omega^*(M)$ since it contains $\mathcal{C}^\infty_c\Omega^*(M)$, and we point out that it is a closed domain for d. Indeed, if (v, w) is in the closure of the graph of $(d, \mathscr{D}(d)')$ so that we have

$$(v_n, dv_n) \in \mathscr{D}(d)' \times \mathscr{D}_{\max}(d) \text{ s.t. } v_n \to v, \ dv_n \to w$$

then $w = dv$ because $\mathscr{D}_{\max}(d)$ is a closed domain and

$$\int_M d(u \wedge v) = \int_M du \wedge v \pm u \wedge dv = \lim \int_M du \wedge v_n \pm u \wedge dv_n = \lim \int_M d(u \wedge v_n) = 0;$$

thus $v \in \mathscr{D}(d)'$ as required. Finally if $(\mathscr{D}(d), d)$ make up a Hilbert complex, i.e., if $d(\mathscr{D}(d)) \subseteq \mathscr{D}(d)$, then so do $(\mathscr{D}(d)', d)$ since whenever $u \in \mathscr{D}(d)$ we have

$$\int_M d(u \wedge v) = 0 \implies \int_M d(u \wedge dv) = \pm \int_M d(du \wedge v) = 0.$$

In this way, to each choice of ideal boundary conditions $(\mathscr{D}(d), d)$ we have assigned a 'dual' choice of ideal boundary conditions $(\mathscr{D}(d)', d)$ through the intersection pairing.

We say that a choice of ideal boundary conditions is **self-dual** if

$$\mathscr{D}(d) = \mathscr{D}(d)'.$$

In this case, the intersection pairing descends to cohomology into a non-degenerate pairing and we say that the complex **satisfies Poincaré duality.** A manifold has *negligible boundary* or *satisfies the L^2-Stokes theorem* precisely when the maximal domain is self-dual.

As mentioned in the first lecture, every Hilbert complex has an adjoint complex. Let us see how the dual complex and the adjoint complex are related.

Recall that for every choice of ideal boundary conditions $(\mathscr{D}(d), d)$ there is a Hilbert complex $(\mathscr{D}(\delta), \delta)$ with

$$\mathscr{D}(\delta) = \{v \in \mathscr{D}_{\max}(\delta) : \langle du, v\rangle_{L^2} = \langle u, \delta v\rangle_{L^2} \text{ for all } u \in \mathscr{D}(d)\}.$$

A useful framework for thinking about this is the 'boundary pairing',

$$\begin{array}{rcl} \mathscr{D}_{\max}(d) \times \mathscr{D}_{\max}(\delta) & \xrightarrow{G_d} & \mathbb{R} \\ (u, v) & \longmapsto & \langle du, v\rangle_{L^2} - \langle u, \delta v\rangle_{L^2} \end{array} .$$

The adjoint domain of $\mathscr{D}(d)$ is the annihilator of $\mathscr{D}(d)$ with respect to G_d,

$$\mathscr{D}(\delta) = (\mathscr{D}(d))^{\perp_{G_d}}.$$

If we write out the integrals in G_d we get

$$\langle du, v\rangle_{L^2} - \langle u, \delta v\rangle_{L^2} = \int_M du \wedge *v - \int_M u \wedge *\delta v = \int_M du \wedge *v \pm \int_M u \wedge d * v = \int_M d(u \wedge *v)$$

and comparing this to the discussion above we see that

$$\mathscr{D}(d)' = *\mathscr{D}(\delta).$$

We will see below how to obtain a self-dual domain on spaces with conic singularities by considering the possible self-adjoint domains for $d + \delta$.

The second reason we want to emphasize is *identifying reduced L^2-cohomology for complete metrics*. For a complete metric, $d + \delta$ is essentially self-adjoint by Gaffney's Theorem[29] and so L^2-cohomology is automatically self-dual. Unfortunately it is often infinite dimensional because the exterior derivative fails to have closed range. As explained in the first lecture, when this happens we study the reduced L^2-cohomology and this coincides with a vector space of harmonic forms. These spaces continue to satisfy Poincaré duality. Indeed, this follows from the fact that if ω is closed and coclosed, then so is $*\omega$.

[29]Theorem 1.3 above.

Remark 3. *Note that when working with weighted L^2-spaces, the intersection pairing is a pairing between $x^a L^2\Omega^*(M)$ and $x^{-a}L^2\Omega^*(M)$. This leads not to Poincaré duality but to a 'generalized' Poincaré duality analogous to the duality between the minimal and maximal L^2 complexes.*

3.5. Two facts from the b-calculus. [30].

The b-calculus was discussed in the lecture series by Daniel Grieser. We briefly recall two important facts upon which we build our discussion of Hodge cohomology.

Recall that a b-differential operator is a differential operator that in local coordinates can be written as a polynomial in vector fields tangent to the boundary. Thus if M is a manifold with boundary, x is a boundary defining function and z represents local coordinates on ∂M, then a b-differential operator of order k, P, has the form

$$P = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,z)(x\partial_x)^j \partial_z^\alpha.$$

A complex number s is an indicial root of P if

$$I(P;s) = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(0,z)(s)^j \partial_z^\alpha$$

is not invertible on ∂M.

Proposition 3.1. *As an unbounded operator on $x^a L^2$, an elliptic b-differential operator is Fredholm if and only if a is not an indicial root.*

Proposition 3.2. *Let $P \in \mathrm{Diff}_b^k(M;E)$ be an elliptic b-differential operator acting on sections of vector bundle. If $u \in x^\ell L^2(M;E)$ is such that $Pu = v' + w'$ with v' polyhomogeneous and $w' \in x^{\ell'}L^2(M;E)$ with $\ell' > \ell$ then $u = v + w$ with v polyhomogeneous and $w \in x^{\ell'-}H^k(M;E)$.*

The error term is in $x^{\ell'}H^k(M;E)$ if ℓ' does not correspond to an indicial root.

3.6. Reduced L^2-cohomology of a b-metric. [31]

As an example of computing reduced L^2-cohomology through Hodge cohomology. Let us return to the d metric from the second lecture and then further simplify to a b-metric. Thus M is the interior of a manifold with boundary and, in a collar neighborhood of ∂M, the metric takes the form

$$g_b = \frac{dx^2}{x^2} + g_{\partial M}.$$

We worked out the $x^\alpha L^2$ cohomology for this metric under the assumption that $-\alpha \notin \{0, \ldots, m\}$ and pointed out that we were making this assumption to guarantee that the image of the exterior derivative is closed. Our discussion of Poincaré duality shows that the case $\alpha = 0$ is particularly interesting.

[30]For a full discussion see The Atiyah-Patodi-Singer index theorem by Richard Melrose

[31]This is already treated in The Atiyah-Patodi-Singer index theorem by Richard Melrose. See 'Hodge cohomology of gravitational instantons' by Tamas Hausel, Eugénie Hunsicker, and Rafe Mazzeo for a treatment that includes fibered boundary and fibered cusp metrics.

There is a vector bundle, known as the b-tangent bundle and denoted ${}^bTM$, whose sections 'are' the vector fields on $\overline{M}$ that are tangent to the boundary, $\mathcal{V}_b$. Precisely the Serre-Swan theorem[32] gives us a vector bundle and bundle map

$$i : {}^bTM \longrightarrow T\overline{M}$$

such that $i_*\mathcal{C}^\infty(\overline{M}; {}^bT\overline{M}) = \mathcal{V}_b \subseteq \mathcal{C}^\infty(\overline{M}; T\overline{M})$.

In local coordinates near the boundary ${}^bTM$ is spanned by $x\partial_x$ and ∂_z, and the dual vector bundle ${}^bT^*M$, known as the b-cotangent bundle is locally spanned by $\frac{dx}{x}$ and dz. The key fact is that $x\partial_x$, as a section of ${}^bTM$, does not vanish at the boundary and similarly $\frac{dx}{x}$, as a section of ${}^bT^*M$, is not singular at the boundary. Moreover, the map i between the b-tangent bundle and the tangent bundle of $\overline{M}$ is canonically an isomorphism over M. Thus when doing analysis over the interior of M there is no loss in replacing the tangent bundle with the b-tangent bundle, or replacing differential forms with b-differential forms.

The exterior powers of the b-tangent bundle decompose in a collar neighborhood of the boundary as

$$\Lambda^j {}^bT^*M = \Lambda^j T^*\partial M \oplus \frac{dx}{x} \wedge \Lambda^{j-1} T^*\partial M$$

and correspondingly $d + \delta$ takes the form

$$d + \delta = \begin{pmatrix} d_{\partial M} + \delta_{\partial M} & -x\partial_x \\ x\partial_x & -(d_{\partial M} + \delta_{\partial M}) \end{pmatrix}.$$

Crucially, we note that $d + \delta$ is an elliptic b-differential operator.

Theorem 3.3. *If (M, g) is a manifold with a cusp metric then for every j there is a natural isomorphism*

$$\overline{\mathrm{H}}^j_{L^2}(M, g) \longrightarrow \mathrm{Image}\left(\mathrm{H}^j_{x^\varepsilon L^2}(M, g) \longrightarrow \mathrm{H}^j_{x^{-\varepsilon} L^2}(M, g)\right)$$

for any sufficiently small $\varepsilon > 0$.

Proof. Given $\omega \in \mathrm{Ker}_{L^2}(d + \delta)$ we know from Proposition 3.2 that ω has a polyhomogenous expansion and hence is in $x^\varepsilon L^2\Omega^*(M)$ for sufficiently small ε, independent of ω. Thus we have a natural map

$$\Phi : \mathrm{Ker}_{L^2}(d + \delta) \longrightarrow \mathrm{Image}\left(\mathrm{H}^j_{x^\varepsilon L^2}(M, g) \longrightarrow \mathrm{H}^j_{x^{-\varepsilon} L^2}(M, g)\right)$$

and we will show that this is an isomorphism.

If $\Phi(\omega) = 0$ then ω defines a trivial $\mathrm{H}_{x^{-\varepsilon}L^2}(M, g)$ class and so there exists $\zeta \in x^{-\varepsilon}L^2\Omega^j(M)$ such that $\omega = d\zeta$. Without loss of generality ζ is orthogonal to $\mathrm{Ker}_{x^{-\varepsilon}L^2} d$. By the Kodaira decomposition, which holds without taking closures because we know that the weighted L^2 cohomology is finite dimensional for this weight, this means that $\zeta = \delta\zeta'$ for some $\zeta' \in \mathscr{D}_{\max,-\varepsilon}(d))^*$. Thus $(d + \delta)\zeta = \omega$ and, since ω is polyhomogeneous and $d + \delta$ is an elliptic b-differential operator, we get that ζ is polyhomogeneous.

Let us write

$$\omega = \alpha + \frac{dx}{x} \wedge \beta, \quad \zeta = \mu + \frac{dx}{x} \wedge \gamma.$$

[32]For a direct construction of the b-tangent bundle and other 'rescaled' bundles, see §8.1-8.2 of The Atiyah-Patodi-Singer index theorem by Richard Melrose.

Since ω and ζ are polyhomogeneous we know that $|\alpha|$, $|\beta|$ are in $\mathcal{O}(x^c)$ for some $c > \varepsilon$ and $|\mu|$, $|\gamma|$ are in $\mathcal{O}(x^{c'})$ for some $c' > -\varepsilon$. Thus we have

$$\|\omega\|^2_{L^2} = \int_M d\zeta \wedge *\omega = \int_M d(\zeta \wedge *\omega) = \lim_{\delta\to 0} \int_{\{x=\delta\}} \zeta \wedge *\omega = \lim_{\delta\to 0} \int_{\{x=\delta\}} \mu \wedge *\beta$$

which, since the integrand is $\mathcal{O}(\delta^{c+c'})$ and $c + c' > 0$, is equal to zero. This shows that Φ is injective.

To show that Φ is surjective, we take a class in the image of the $x^\varepsilon L^2$ cohomology in $x^{-\varepsilon}L^2$. We can represent this by a form $\eta \in x^\varepsilon L^2\Omega^j(M)$ that is polyhomogeneous (e.g., by using a harmonic representative). Since $d+\delta$ is Fredholm as an operator on $x^{-\varepsilon}L^2\Omega^j$ we can use the natural pairing between $x^\varepsilon L^2\Omega^*(M)$ and $x^{-\varepsilon}L^2\Omega^*(M)$ to establish a decomposition

$$x^{-\varepsilon}L^2\Omega^*(M) = \mathrm{Image}_{x^{-\varepsilon}L^2}(d+\delta) \oplus \mathrm{Ker}_{x^\varepsilon L^2}(d+\delta) = \mathrm{Image}_{x^{-\varepsilon}L^2}(d+\delta) \oplus \mathrm{Ker}_{L^2}(d+\delta).$$

Thus we may write $\eta = (d+\delta)\zeta + \gamma$ with $\zeta \in x^{-\varepsilon}L^2\Omega^*(M)$ and $\gamma \in \mathrm{Ker}_{L^2}(d+\delta)$. To see that $\Phi(\gamma) = [\eta]$ we just need to show that $\delta\zeta = 0$.

We can write $\delta\zeta = \eta - d\zeta - \gamma$ and then, formally,

$$\|\delta\zeta\|^2_{x^{-\varepsilon}L^2} = \|\delta\zeta, \eta - d\zeta - \gamma\|_{x^{-\varepsilon}L^2} = \|\zeta, d(\eta - d\zeta - \gamma)\|_{x^{-\varepsilon}L^2} = 0.$$

What is formal about this argument is that we ignored potential boundary terms from the integration by parts. Another argument using polyhomogeneity shows that these boundary terms vanish. □

3.7. **A choice of self-dual boundary condition for an isolated conic singularity.** [33]

As an example of making a choice of domain using Hodge cohomology, let us return to the wedge metric from the second lecture and then further simplify to a conic metric. Thus M is the interior of a manifold with boundary and in a collar neighborhood of ∂M the metric takes the form

$$g_c = dx^2 + x^2 g_{\partial M}.$$

Consider the decomposition of the de Rham operator near the boundary. First, with respect to the splitting

$$\Omega^j M = \Omega^j(\partial M) \oplus dx \wedge \Omega^{j-1}(\partial M),$$

we have

$$d + \delta = \begin{pmatrix} d_{\partial M} + \frac{1}{x^2}\delta_{\partial M} & -\frac{1}{x}(v+1-2j) - \partial_x \\ \partial_x & -d_{\partial M} - \frac{1}{x^2}\delta_{\partial M} \end{pmatrix}.$$

We get a much more symmetric expression if we consider instead a weighted splitting

$$x^j\Omega^j(\partial M) \oplus dx \wedge x^{j-1}\Omega^{j-1}(\partial M)$$

for which we get

$$d + \delta = \begin{pmatrix} \frac{1}{x}(d_{\partial M} + \delta_{\partial M}) & -\frac{(v-j)}{x} - \partial_x \\ \frac{j}{x} + \partial_x & -\frac{1}{x}(d_{\partial M} + \delta_{\partial M}) \end{pmatrix}. \tag{3.1}$$

Significantly, the model operator for the de Rham operator is the de Rham operator of the boundary.

[33] Most of the results in this section are in 'On the spectral geometry of spaces with cone-like singularities' by Jeff Cheeger. For more on geometric analysis on similar spaces see the papers by Juan Gil, Thomas Krainer and Gerardo Mendoza and the references at the end of this section.

This scaling is naturally carried out by rescaling the cotangent bundle to the wedge cotangent bundle. Let

$$\mathcal{V}^*_{\mathrm{w}} = \{\omega \in \mathcal{C}^\infty(\overline{M}; T^*\overline{M}) : i^*_{\partial M}\omega = 0\}.$$

By the Serre-Swan theorem there is a vector bundle, the **wedge cotangent bundle**, ${}^{\mathrm{w}}T^*M$, over $\overline{M}$ equipped with a bundle map $i : {}^{w}T^*M \longrightarrow T^*\overline{M}$ such that

$$i_*\mathcal{C}^\infty(\overline{M}; {}^{\mathrm{w}}T^*M) = \mathcal{V}^*_{\mathrm{w}} \subseteq \mathcal{C}^\infty(\overline{M}; T^*\overline{M}).$$

The map i is an isomorphism over the interior of M, which is where we do our analysis, so there is no loss in replacing the cotangent bundle with the wedge cotangent bundle. Note that, near the boundary,

$$i_*\mathcal{C}^\infty(\overline{M}; \Lambda^j({}^{w}T^*M)) = x^j\Omega^j(\partial M) \oplus dx \wedge x^{j-1}\Omega^{j-1}(\partial M),$$

so we achieve the scaling we wanted by working with **wedge differential forms**.

Having made this change note that $x(d+\delta)$ is a b-differential operator on wedge differential forms. This allows us to apply the analysis from the previous sections. To start with, directly from the definition of the maximal domain we have that

$$u \in \mathcal{D}_{\max}(d+\delta) \implies u \in L^2\Omega^*(M), \text{ and } x(d+\delta)u \in xL^2\Omega^*(M)$$

to which we can apply Proposition 3.2.

Lemma 3.4. *If $u \in \mathcal{D}_{\max}(d+\delta)$ then u has a partial asymptotic expansion*[34]

$$u(x,z) = \sum_{\substack{s_j \in \mathrm{spec}_b(x(d+\delta)) \\ s_j \in (-\frac{v}{2}-\frac{1}{2}, -\frac{v}{2}+\frac{1}{2})}} u_{s_j}(z)x^{s_j} + \widetilde{u}(x,z)$$

with $\widetilde{u} \in x^{1-}L^2\Omega^j(M)$.

Remark 4. *As an exercise work out the indicial roots of $x(d+\delta)$ from* (3.1). *For symmetry you may want to consider $x^{v/2}(x(d+\delta))x^{-v/2}$.*

Using the boundary pairing it is not hard to see that

$$u \in \mathscr{D}_{\min}(d+\delta) \iff u \in \mathscr{D}_{\max}(d+\delta) \text{ and } u_{s_j} = 0 \text{ for all } s_j \text{ in the sum above.}$$

Thus we can identify $\mathscr{D}_{\max}(d+\delta)/\mathscr{D}_{\min}(d+\delta)$, the space of all domains in between the maximal and minimal ones, with the set of coefficients in the partial asymptotic expansion. These coefficients are the 'Cauchy data' at our high codimension boundary.

Among this Cauchy data is a topological one[35],

$$u_{\frac{v}{2}}(z) = A(u) + dx \wedge B(u), \quad A, B \in \mathcal{H}^{v/2}_{L^2}(\partial M).$$

All of the other indicial roots occurring in the partial asymptotic expansion above corresponding to 'small' eigenvalues of $(d_{\partial M} + \delta_{\partial M})$. If we allow ourselves to scale the metric $g_{\partial M}$, we can push all of these small eigenvalues out of this interval.

[34]The interval occurring in this sum is imposed on us by the volume form which has the form $x^v\, dx\, \mathrm{d}v_{\partial M}$. The expansion is that part of the polyhomogeneous expansion that the indicial roots would give that lies in L^2 but not in xL^2 (since that gets absorbed by the error term). The error term has better regularity, it is in the b-Sobolev space of order one.

[35]In including A, B in $\mathrm{H}^{v/2}(\partial M)$ we are eliding a weight coming from the fact that we have a wedge differential form.

Proposition 3.5. *For a suitably scaled metric conic metric, $g_c = dx^2 + x^2 g_{\partial M}$, the space of Cauchy data for $d+\delta$ coincides with two copies of the cohomology of the boundary in degree $v/2$. In particular, $d+\delta$ is essentially self-adjoint if and only if*

$$\mathrm{H}^{v/2}(\partial M) = \{0\}. \tag{3.2}$$

It follows that this condition characterizes when the L^2 Stokes' theorem holds for conic metrics (without any scaling necessary).

The reason why there is no scaling necessary for the final statement is that L^2 Stokes' theorem holds precisely when $\mathscr{D}_{\min}(d) = \mathscr{D}_{\max}(d)$ and neither of these notice the scaling of the metric on the boundary, as this yields quasi-isometric metrics.

A space satisfying the topological condition (3.2) is known as a **Witt space**. If the space is not Witt then $\mathscr{D}_{\min}(d) \neq \mathscr{D}_{\max}(d)$ and we are interested in finding a self-dual domain.

As this is related to finding a self-adjoint domain for $d+\delta$, let us consider the boundary pairing for this operator

$$\begin{aligned} \mathscr{D}_{max}(d+\delta) \times \mathscr{D}_{\max}(d+\delta) &\xrightarrow{\quad G_{d+\delta} \quad} \mathbb{R} \\ (u,w) &\longmapsto \langle (d+\delta)u, w\rangle - \langle u, (d+\delta)w\rangle \end{aligned}$$

which, using polyhomogeneity and Stokes' theorem, is given by

$$G_{d+\delta}(u,w) = \langle A(u), B(w)\rangle_{\partial M} - \langle B(u), A(w)\rangle_{\partial M}.$$

Following Cheeger, we choose a subspace $V_a \subseteq \mathcal{H}^{v/2}_{L^2}(\partial M)$, denote the orthogonal complement by $V_r \subseteq \mathcal{H}^{v/2}_{L^2}(\partial M)$, and then define a domain for $d+\delta$ by

$$\mathscr{D}_{V_a}(d+\delta) = \{u \in \mathscr{D}_{\max}(d+\delta) : A(u) \in V_a \text{ and } B(u) \in V_r\}.$$

Note that

$$w \in \mathscr{D}_{\max}(d+\delta) \text{ is s.t. } G_{d+\delta}(u,w) = 0 \text{ for all } u \in \mathscr{D}_{V_a}(d+\delta) \iff w \in \mathscr{D}_{V_a}(d+\delta),$$

or stated differently:

Proposition 3.6. *For any subspace $V_a \subseteq \mathcal{H}^{v/2}_{L^2}(\partial M)$, the operator $(d+\delta, \mathscr{D}_{V_a}(d+\delta))$ is self-adjoint.*

Now we want to turn this into a domain for d. Given $u \in \mathscr{D}_{\max}(d)$ we define

$$u_\delta = \text{orthogonal projection of } u \text{ onto } \overline{\delta(\mathscr{D}_{\max}(\delta))}$$

Since $\overline{\delta(\mathscr{D}_{\max}(\delta))}^{\perp} = \ker(d, \mathscr{D}_{\min}(d))$, we have $u - u_\delta \in \mathscr{D}_{\min}(d)$ and hence $u_\delta \in \mathscr{D}_{\max}(d)$. Even better u_δ is in $\mathscr{D}_{\max}(d+\delta)$ so u_δ has a partial asymptotic expansion and we define

$$\mathscr{D}_{V_a}(d) = \{u \in \mathscr{D}_{\max}(d) : A(u_\delta) \in V_a\}.$$

We can similarly take any $w \in \mathscr{D}_{\max}(\delta)$, project it orthogonally onto $\overline{d(\mathscr{D}_{\max}(d))}$, and the resulting form w_d will be in $\mathscr{D}_{\max}(\delta) \cap \mathscr{D}_{\max}(d+\delta)$ and in particular it will have a partial asymptotic expansion. It is easy to see that

$$G_d(u,w) = G_d(u_\delta, v_d)$$

and then since u_δ and v_d have partial polyhomogeneous expansions we can compute this as

$$G_d(u,w) = \langle A(u_\delta), B(w_d)\rangle_{\partial M}.$$

In particular it follows that the adjoint domain of $\mathscr{D}_{V_a}(d)$ is

$$\mathscr{D}_{V_r}(\delta) = \{u \in \mathscr{D}_{\max}(\delta) : B(w_d) \in V_r\}$$

and hence

$$\mathscr{D}_{V_a}(d+\delta) = \mathscr{D}_{V_a}(d) \cap \mathscr{D}_{V_r}(\delta) = \mathscr{D}_{V_a}(d) \cap \mathscr{D}_{V_a}(d)^*.$$

Thus each subspace $V_a \subseteq \mathcal{H}^{v/2}(\partial M)$ yields a Hilbert complex $\mathscr{D}_{V_a}(d)$ whose associated de Rham operator is $(d+\delta, \mathscr{D}_{V_a}(d+\delta))$. When will this satisfy Poincaré Duality? From the discussion above we want

$$\mathscr{D}_{V_a}(d) = \mathscr{D}_{V_a}(d)' = *\mathscr{D}_{V_a}(d)^* = *\mathscr{D}_{V_r}(\delta),$$

so from the description of V_a and V_r this happens precisely when

$$V_a = *V_a^\perp.$$

The existence of such a subspace is equivalent to the vanishing of the signature of ∂M. We refer to such a choice as **Cheeger ideal boundary conditions** and to a space where such boundary conditions exist as a **Cheeger space.**

Theorem 3.7. *A space with isolated conic singularities admits Cheeger ideal boundary conditions if and only if the signature of its boundary (i.e., the signature of the link of the conic singularity) vanishes.*

It turns out[36] that the signature of the resulting Hilbert complex is independent of the choice of Cheeger ideal boundary conditions, and that this signature is invariant under stratified homotopy equivalences and Cheeger space bordisms.

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[36]See the papers by P.A., Eric Leichtnam, Rafe Mazzeo, and Paolo Piazza; also the work of these authors with Markus Banagl for a topological interpretation and the paper 'On the Hodge theory of stratified spaces' by P.A. for a survey.