Supersymmetric gauge theory, monopole moduli space, and 'no-exotics' as a generalized Sen conjecture

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ABSTRACT: Recent developments in quantum supersymmetric gauge theory have implications for the L^2 cohomology of families of (twisted) Dirac-Dolbeault operators on monopole moduli space. Wall-crossing formulae for BPS states lead to predictions for where the Dirac operators fail to be Fredholm, and how their kernels jump, as parameters of the family are varied. The no-exotics property of BPS states leads to a generalization of the Sen conjecture: All non-trivial L^2 cohomology of the Dolbeault operators is concentrated in the middle anti-holomorphic degree. The aim of these lectures is to give a pedagogical description of how these predictions arise from supersymmetric gauge theory.

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Contents

1 Lecture I: Classical gauge theory and monopoles

In this lecture we summarize some properties of (singular) monopole moduli spaces that will be important for subsequent developments. We show how these spaces arise in the description of BPS field configurations in classical $\mathcal{N} = 2$ supersymmetric Yang–Mills theory.

1.1 Moduli space of (singular) monopoles

Let G be a compact simple Lie group. Consider Yang–Mills–Higgs theory on $\mathbb{R}^{1,3}$ with action

$$S_{\text{ymh}} = -\frac{1}{g_0^2}\int_{\mathbb{R}^{1,3}} [(F, \star F) + (DX, \star DX)] \ . \tag{1.1}$$

Here g_0 is the 'bare' Yang–Mills coupling. It plays no role in this lecture but will be important in the next. $(\ ,\ )$ denotes a Killing form, normalized such that the squared-length of long roots is two.[1] F is the curvature (fieldstrength) of a connection A (gauge field), $F = \mathrm{d}A + A \wedge A$, on a principal G-bundle $\mathcal{P} \to \mathbb{R}^{1,3}$.[2] X is a section of the adjoint

[1]It is given in terms of the Cartan–Killing form by $(A, B) = -\frac{1}{2h^\vee}\,\mathrm{tr}(\,\mathrm{ad}(A)\,\mathrm{ad}(B))$, where $h^\vee$ is the dual Coxeter number.

[2]G-bundles over $\mathbb{R}^{1,3}$ are topologically trivial, but later we will remove a line $\mathbb{R}_t \times \{\vec{0}\}$ from $\mathbb{R}_t \times \mathbb{R}^3$ and then the G-bundle $\mathcal{P}$ over the resulting space might be topologically nontrivial.

bundle (Higgs field), and $DX = \mathrm{d}X + [A, X]$ is the covariant derivative in the adjoint representation.

Magnetic monopoles are static solutions of the Yang–Mills–Higgs equations which additionally solve the Bogomolny equations on $\mathbb{R}^3$,

$$F = \star_3 DX \ , \tag{1.2}$$

and satisfy the following boundary conditions. We choose a regular element[3] X_∞ of the Lie algebra $\mathfrak{g}$ of G. This selects a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, as well as a system of simple roots $\alpha_I \in \mathfrak{t}^\vee$ and simple co-roots $H_I \in \mathfrak{t}$, where $I = 1, \ldots, \mathrm{rnk}\,\mathfrak{g}$. With respect to this system, X_∞ lies in the fundamental Weyl chamber of $\mathfrak{t}$. The integer span of the simple roots is the root lattice $\Lambda_{\mathrm{rt}} \subset \mathfrak{t}^\vee$, and the integer space of the simple co-roots in the co-root lattice $\Lambda_{\mathrm{cr}} \subset \mathfrak{t}$. Then, in a suitable gauge, we have

$$(\mathrm{bc}_\infty): \qquad X = X_\infty - \frac{q_{\mathrm{m}}}{2r} + \cdots \ , \qquad F = \frac{q_{\mathrm{m}}}{2}\omega + \cdots \ , \qquad r \to \infty \ . \tag{1.3}$$

Here q_{m} is the magnetic charge, and $\omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ is the volume form on the two-sphere with (r, θ, ϕ) the standard spherical coordinates on $\mathbb{R}^3$. Single-valuedness of the transition function for the bundle on two-spheres at sufficiently large radius, and contractibility of these two-spheres in $\mathbb{R}^3$, implies that $q_{\mathrm{m}} \in \Lambda_{\mathrm{cr}}$. In other words $q_{\mathrm{m}} = \sum_I n_{\mathrm{m}}^I H_I$ for some integers n_{m}^I. It is known that solutions exist iff all $n_{\mathrm{m}}^I \geq 0$ and at least one is positive [1].

We are also interested in singular magnetic monopoles; that is, we allow singularities in the monopole field corresponding, physically, to the introduction of 't Hooft line defects. For simplicity, we will consider a single line defect at the origin in $\mathbb{R}^3$. The data of the singularity is given by an element P of the co-character lattice $\Lambda_G \cong \mathrm{Hom}(\mathrm{U}(1), T)$, where T is the Cartan torus of G with Lie algebra $\mathfrak{t}$. [4] Singular monopoles are solutions to (1.2), satisfying the boundary conditions (1.3) as $r \to \infty$, and the boundary conditions

$$(\mathrm{bc}_0): \qquad X = -\frac{P}{2r} + O(r^{-1/2}) \ , \qquad F = \frac{P}{2}\omega = O(r^{-3/2}) \ , \qquad \text{as} \quad r \to 0 \ . \tag{1.4}$$

Physically, P specifies the embedding of a Dirac monopole (*i.e.* U(1) monopole) into the non-abelian gauge group. This type of singularity is electromagnetically dual to a Wilson line defect, which corresponds to a point source for the electric field in $\mathbb{R}^3$. 't Hooft defects were first introduced in [2] as a tool for studying phases of quantum Yang–Mills theory. We follow the conventions of [3]. Singular monopoles of this type were first considered in the mathematics literature by Kronheimer [4]. A few further references include [5–7]. In the presence of singularities, the asymptotic magnetic charge q_{m} need no longer sit in the co-root lattice, but will in general sit in a shifted copy of the co-root lattice: $q_{\mathrm{m}} \in P + \Lambda_{\mathrm{cr}}$.

[3] Thus we are restricting to the case of 'maximal symmetry breaking' where the centralizer of X_∞ is a Cartan torus.

[4] We have $\Lambda_{\mathrm{cr}} \subseteq \Lambda_G \subseteq \Lambda_{\mathrm{rt}}^\vee$, where $\Lambda_{\mathrm{rt}}^\vee$ is the integral dual of the root lattice, sometimes referred to as the magnetic weight lattice. At the two extremes, $\Lambda_{\mathrm{cr}} \cong \Lambda_G$ when G has trivial center, and $\Lambda_G \cong \Lambda_{\mathrm{rt}}^\vee$ when G is simply-connected. In general, $\Lambda_G/\Lambda_{\mathrm{cr}} \cong \mathcal{Z}(G)$, the center of G, while $\Lambda_{\mathrm{rt}}^\vee/\Lambda_G \cong \pi_1(G)$.

Define the *relative magnetic charge* by

$$\tilde{q}_{\rm m} := q_{\rm m} - P^- \ , \tag{1.5}$$

where P^- is the representative of the Weyl orbit of P in the closure of the anti-fundamental Weyl chamber: $\langle \alpha_I, P^- \rangle \leq 0$, $\forall \alpha_I$, where $\langle \ , \ \rangle : \mathfrak{t}^\vee \times \mathfrak{t} \to \mathbb{R}$ is the natural pairing. This is a generalization of Kronheimer's 'non-abelian' charge. We have $\tilde{q}_{\rm m} \in \Lambda_{\rm cr}$, and hence $\tilde{q}_{\rm m} = \sum_I \tilde{n}_{\rm m}^I H_I$ for some integers $\tilde{n}_{\rm m}^I$. We conjectured in [8] that singular monopole solutions exist iff $\tilde{n}_{\rm m}^I \geq 0$, $\forall I$, and provided evidence for this in [9] using a string theory brane construction. This is motivated by the same type of interpretation that Weinberg [10] gave for ordinary monopoles in terms of fundamental constituents. Existence results for singular monopoles along these lines but in a slightly different context have been obtained in [11].

We are only interested in solutions to the Bogomolny equation up to gauge-equivalence. We distinguish between the group of gauge transformations, $\mathcal{G} = {\rm Aut}(\mathcal{P})$, and the group of local gauge transformations, $\mathcal{G}^0$. The latter is a subgroup of the group of gauge transformations, consisting of those transformations that approach the identity as $r \to \infty$. Additionally, in the case that a defect is present, we require both $\mathcal{G}, \mathcal{G}^0$ to leave the charge P invariant. The moduli space of singular monopoles is defined as

$$\overline{\underline{\mathcal{M}}}(P; q_{\rm m}, X_\infty) := \{(A,X) \mid F = \star_3 DX \ \& \ ({\rm bc}_\infty) \ \& \ ({\rm bc}_0)\} \Big/ \mathcal{G}^0 \ . \tag{1.6}$$

When there are no singularities we get the ordinary (Euclidean) monopole moduli spaces,

$$\mathcal{M}(q_{\rm m}, X_\infty) := \{(A,X) \mid F = \star_3 DX \ \& \ ({\rm bc}_\infty)\} \Big/ \mathcal{G}^0 \ . \tag{1.7}$$

These spaces have a number of remarkable properties that will be important in the following:

- Each admits a hyperkähler metric (away from singular loci in the case of $\overline{\underline{\mathcal{M}}}$). This is most easily seen by observing that $\overline{\underline{\mathcal{M}}}, \mathcal{M}$ can be defined as a hyperkähler quotient of the space of field configurations (A, X), endowed with the flat metric, by the group of local gauge transformations. The Bogomolny equations are the moment maps in this construction. We will denote the hyperkähler metrics by $g_{\overline{\underline{\mathcal{M}}}}$, $g_{\mathcal{M}}$.
- The ordinary monopole moduli spaces $\mathcal{M}$ are smooth and complete [12]. The singular monopole moduli spaces can (but do not necessarily) have singularities on co-dimension 4 loci. The presence of singularities is related to the phenomenon of monopole bubbling [7]. When the charge of the singularity is minimal—in the sense that its Weyl orbit forms the complete set of weights of a representation of the Langlands dual group $G^\vee$—there is no monopole bubbling and the moduli space is smooth.[5] In the simplest case the singularities are of orbifold type. It is expected

[5]For example, a 't Hooft charge of half the simple co-root, $P = \frac{1}{2}H$, in the $G = {\rm SO}(3)$ theory is minimal: Its Weyl orbit gives the weights of the two-dimensional representation of ${\rm SO}(3)^\vee = {\rm SU}(2)$. This is the only minimal 't Hooft charge in either the SO(3) or SU(2) theory.

that singular loci of $\overline{\mathcal{M}}$ correspond to $\overline{\mathcal{M}}$'s of lower dimension, such that there can be a nested sequence of singular loci within singular loci. As far as I know, a complete picture of the structure of these singularities for general gauge groups with general 't Hooft charges P has not been developed.

- The dimension of $\overline{\mathcal{M}}(P; \gamma_{\rm m}, X_\infty)$ was computed in [8] by modifying the Callias index theorem [13] to account for the possibility of singularities. It takes the form

$$\dim_{\mathbb{R}} \overline{\mathcal{M}}(P; q_{\rm m}, X_\infty) = 4 \sum_I \tilde{n}^I_{\rm m} \ , \tag{1.8}$$

  in terms of the components of the relative magnetic charge. This builds on earlier results of [5, 7] for singular monopoles. Furthermore, when defects are absent, it reduces to the well-known result for ordinary monopoles in terms of the components of the asymptotic magnetic charge [10, 14].

- These moduli spaces carry a number of isometries. The Lie algebras of Killing fields are

$$\begin{aligned} \mathcal{M}: &\qquad \mathbb{R}^3 \oplus \mathfrak{so}(3) \oplus \mathfrak{t} \ , \\ \overline{\mathcal{M}}: &\qquad \mathfrak{so}(3) \oplus \mathfrak{t} \ . \end{aligned} \tag{1.9}$$

  The factor of $\mathbb{R}^3$ corresponds to the fact that spatial translation of a solution gives another solution, when defects are absent. However the presence of a defect breaks translational symmetry. The $\mathfrak{so}(3)$ likewise originates from spatial rotations, (where the fixed point of the rotation is the the location of the singularity in the case of singular monopoles).

  The factor of the Cartan subalgebra originates from the action of asymptotically non-trivial gauge transformations on solutions to the Bogomolny equation. These gauge transformations can be identified with elements of $\mathcal{G}/\mathcal{G}^0$ that leave the asymptotic data invariant. Hence they are gauge transformations that asymptote, as $r \to \infty$ to an element of T, the Cartan torus. The induced isometries of $\overline{\mathcal{M}}, \mathcal{M}$ will be especially important in the following, as they are related to electric charge. In addition to preserving the metric, they preserve the hyperkähler structure, and hence are generated by tri-holomorphic Killing fields. We define a Lie algebra homomorphism

$$\mathrm{G} : \mathfrak{t} \to \mathfrak{isom}_{\mathbb{H}}(\overline{\mathcal{M}}) \qquad (\text{or } \mathfrak{isom}_{\mathbb{H}}(\mathcal{M})) \ , \tag{1.10}$$

  as follows. Given a point $[(A, X)] \in \overline{\mathcal{M}}$ or $\mathcal{M}$, and an element $h \in \mathfrak{t}$, find the section of the adjoint bundle, ϵ, that solves $D^2\epsilon + [X, [X, \epsilon]] = 0$ and has $\lim_{r\to\infty} \epsilon = h$.[6] Then the Killing vector $\mathrm{G}(h)$ has directional derivative at $[(A, X)]$ given by $\frac{d}{ds}(A, X) = (-D\epsilon, [\epsilon, X])$.

  The infinitesimal motion generated by the Killing vectors $\mathrm{G}(h)$ exponentiates to a torus action by hyperholomorphic isometries on the moduli space. We will make

[6] The operator acting on ϵ is negative-definite, and one can use the boundary conditions (1.3) and (1.4) to show that the solution is unique.

the simplifying assumption that all components of the (relative) magnetic charge are strictly positive, so that the torus action is effective. Since gauge transformations act on (A, X) in the adjoint representation, it is the magnetic weights, $h \in \Lambda_{\rm mw} = \Lambda_{\rm rt}^{\vee}$, for which $\exp[2\pi \mathrm{G}(h)]$ gives the kernel of this action.

A distinguishing feature of the ordinary moduli spaces is that the Riemannian metric is reducible. More precisely, the universal cover, $\widetilde{\mathcal{M}}$, is metrically a product,

$$\widetilde{\mathcal{M}}(q_{\rm m}, X_\infty) = \mathbb{R}^4 \times \mathcal{M}_0(q_{\rm m}, X_\infty) \ , \tag{1.11}$$

where $\mathbb{R}^4$ carries a flat metric and $\mathcal{M}_0$ is known as the strongly-centered moduli space, in the terminology of [15]. The latter is an irreducible, simply-connected hyperkähler manifold. The moduli space $\mathcal{M}$ is a quotient of $\widetilde{\mathcal{M}}$ by the group of deck transformation $\mathbb{D} \cong \pi_1(\mathcal{M}) \cong \mathbb{Z}$:

$$\mathcal{M}(\gamma_{\rm m}, X_\infty) = \mathbb{R}^3 \times \frac{\mathbb{R} \times \mathcal{M}_0(\gamma_{\rm m}, X_\infty)}{\mathbb{D}} \ . \tag{1.12}$$

Here the $\mathbb{R}^3$ factor is generated by the Killing vectors associated with the $\mathbb{R}^3$ translational isometries, and the $\mathbb{R}$ factor is generated by $\mathrm{G}(X_\infty)$.

An important point for us is that the group of hyperholomorphic isometries acting on the universal cover, and defined by $\mathbb{D}_{\rm g} := \{\exp[2\pi \mathrm{G}(h)] \mid h \in \Lambda_{\rm mw}\}$, is in general a proper subgroup of $\mathbb{D}$. Let ϕ be the isometry generating $\mathbb{D}$. Then, based on the rational map formulation of monopole moduli spaces [16–18], we show in [19] that for any $h \in \Lambda_{\rm mw}$,

$$\exp[2\pi \mathrm{G}(h)] = \phi^{\mu(h)} \ , \tag{1.13}$$

where $\mu : \Lambda_{\rm mw} \to \mathbb{Z}$ is the homomorphism given by $\mu(h) = (q_{\rm m}, h) = \langle q_{\rm m}^{\vee}, h \rangle$. The image of μ is the subgroup $k\mathbb{Z} \subset \mathbb{Z}$, where k is the gcd of the components of $q_{\rm m}^{\vee}$ along the basis of simple roots.

We note that, when $\operatorname{rnk} \mathfrak{g} > 1$, a generic X_∞ generates an irrational curve in T, and $\mathrm{G}(X_\infty)$ generates an irrational direction in the torus of hyperholomorphic isometries. Hence there is no subgroup of $\mathbb{D}$ that acts only on the $\mathbb{R}$ factor in (1.12), and one cannot write $(\mathbb{R} \times \mathcal{M}_0)/\mathbb{D}$ as the quotient of $(S^1 \times \mathcal{M}_0)$ by a cyclic group. For charge k monopoles in the case of $\mathfrak{g} = \mathfrak{su}(2)$, however, $\mathbb{D}_{\rm g}$ acts entirely on the $\mathbb{R}$ factor and we get the simplification $(\mathbb{R} \times \mathcal{M}_0)/\mathbb{D} = (S^1 \times \mathcal{M}_0)/\mathbb{Z}_k$, where $\mathbb{Z}_k = \mathbb{Z}/(k\mathbb{Z}) = \mathbb{D}/\mathbb{D}_{\rm g}$.

1.1.1 Examples

Some examples of ordinary monopole moduli spaces are

$$\{n_{\rm m}^I\} = \begin{cases} \{1\} \ , & \mathcal{M}_0 = \{\mathrm{pt}\} \ , \\ \{2\} \ , & \mathcal{M}_0 = \text{ (double cover of) the Atiyah–Hitchin manifold [12]} \ , \\ \{1,1\} \ , & \mathcal{M}_0 = \text{Taub-NUT manifold [20, 21]} \ , \\ \{2,1\} \ , & \text{see Houghton–Irwin–Mountain [22]} \ , \\ \{1,\ldots,1\} \ , & \text{see Gibbons–Manton [23], Lee–Weinberg–Yi [24], Murray [25].} \end{cases} \tag{1.14}$$

Regarding the last one, Gibbons and Manton determined the approximate form of the metric for general charge k $\mathfrak{su}(2)$ monopoles in an asymptotic region of moduli space where all fundamental constituents are well separated. This metric develops singularities when it is continued into the interior of $\mathcal{M}$. Lee–Weinberg–Yi then showed how a slight modification of this metric (analogous to the relation between the asymptotic form of the Atiyah–Hitchin metric and the Taub–NUT metric) leads to a metric that describes an asymptotic region of $\{1,\ldots,1\}$ monopole moduli space, and which is non-singular if continued into the interior. They conjectured that this metric is the exact metric for $\{1,\ldots,1\}$ monopoles. Murray then proved this via an analysis of the relevant system of Nahm equations.

Explicit examples of moduli spaces of singular monopoles are mostly restricted to $G = \mathrm{SO}(3)$ (or $\mathrm{SU}(2)$) gauge group. In this case let H be the simple co-root, let the 't Hooft charge be $P = \frac{p}{2}H$, and let the relative magnetic charge be $\tilde{q}_{\mathrm{m}} = \tilde{k}H$. Then $\tilde{k}, p \in \mathbb{Z}$ for SO(3) while $p \in 2\mathbb{Z}$ for SU(2). Then we have

$$\{\tilde{k}, p\} = \begin{cases} \{0,p\}\ , & \overline{\mathcal{M}} = \{\mathrm{pt}\}\ , \\ \{1,p\}\ , & \overline{\mathcal{M}} = \text{Taub-NUT}/\mathbb{Z}_{|p|} \text{ (see Cherkis–Kapustin [6])}\ , \\ \{2,\pm 1\}\ , & \text{see Dancer [26]}\ . \end{cases} \tag{1.15}$$

Notice in the second case that when $|p| > 1$ the 't Hooft defect is non-minimal and the moduli space has singularities. The $\mathbb{Z}_{|p|}$ quotient acts on the circle fiber of Taub–NUT, creating an $A_{|p|-1}$ singularity at the nut. It was shown in [22] how the eight-dimensional Dancer manifold arises as a certain infinite-mass limit of the strongly centered moduli space corresponding to the $\{2,1\}$ monopole. Additional examples of $\mathfrak{su}(2)$ singular monopole moduli spaces are discussed in [6, 27, 28].

References not yet included beyond this point

1.2 Embedding into $\mathcal{N} = 2$ super-Yang–Mills

Supersymmetric Yang–Mills theory with $\mathcal{N} = 2$ supersymmetry is the minimal extension of Yang–Mills–Higgs theory, (1.1), in which the Bogomolny equation for monopoles arises as a BPS condition for supersymmetric field configurations. We introduce the theory and explain the meaning of these statements as we go.

The field content is (A, φ, ψ^A), where A is the gauge field as before, and φ is a section of the complexified adjoint bundle whose fibers are $\mathfrak{g}_{\mathbb{C}}$ (a complex Higgs field). The $\{\psi^A\}_{A=1,2}$, are two adjoint-valued Weyl fermions. In other words, let $\mathcal{S}_{\mathrm{D}} \to \mathbb{R}^{1,3}$ be the bundle of Dirac spinors, and let $S_{\mathrm{D}} = \mathcal{S}_+ \oplus \mathcal{S}_-$ with $\mathcal{S}_\pm \to (\mathbb{R}^{1,3})$ is the bundles of positive and negative chirality Weyl spinors over $\mathbb{R}^{1,3}$. The fibers of the latter are copies of $\mathbb{C}^2$. Then $\psi^{1,2} \in \Gamma(\mathcal{S}_+ \otimes \mathrm{ad}(\mathcal{P}))$.[7] The action takes the form

$$S = -\frac{1}{g_0^2}\int_{\mathbb{R}^{1,3}} \left[(F, \star F) + (D\varphi, \star D\bar{\varphi}) - \frac{1}{4}([\varphi,\bar{\varphi}], \star[\varphi,\bar{\varphi}]) + \text{ fermions} \right]\ , \tag{1.16}$$

[7] The "$\mathcal{N} = 2$" refers to the fact that there are two such fermions.

Table 1. Lorentz representations

(j_1, j_2)	name	indices	example
$(1/2, 0)$	(+)-ch. Weyl spinor	$\alpha, \beta = 1, 2$	ψ^A_α
$(0, 1/2)$	(−)-ch. Weyl spinor	$\dot\alpha, \dot\beta = \dot 1, \dot 2$	$\bar\psi_{\dot\alpha A}$
$(1/2, 1/2)$	vector	$\mu, \nu = 0, 1, 2, 3$	A_μ
$(1, 0)$	i.s.d. two-form		$\frac{1}{2}[(1 + i\star)F]_{\mu\nu}$
$(0, 1)$	i.a.s.d. two-form		$\frac{1}{2}[(1 - i\star)F]_{\mu\nu}$

where we have suppressed the terms involving the fermions. Here the bar on $\bar\varphi$ refers to the natural complex conjugation on $\mathfrak{g}_\mathbb{C}$.

In addition to the usual invariance under the Poincaré group of isometries of Minkowski space, this action is invariant under additional symmetries called supersymmetries. These symmetries are generated by a doublet of constant, Granssmann-valued, Weyl spinors, ξ^A, and they map bosons to fermions and vice-versa. To describe this, it is convenient to trivialize the tangent bundle and spinor bundles over $\mathbb{R}^{1,3}$ using canonical bases of sections, and work with the components of fields along such bases. The components form representations of the Lie algebra of the structure group of Minskowski space, $\mathfrak{spin}(1, 3)$. We have $\mathfrak{spin}(1, 3) \cong \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{sl}(2) \times \mathfrak{sl}(2)$, so unitary irreps are labeled by a pair of spins. We have summarized the important representations and corresponding notation in Table 1.

The first two are the two Weyl representations, which are each of dimension two. Note that for $\mathfrak{spin}(1, 3)$, these representations are conjugate to each other and the bar refers to this complex conjugation. Meanwhile for $\mathrm{SU}(2)_R$ the doublet is a pseudo-real representation. We define conjugation to lower the index. This means we take the conjugate spinor to transform in the dual vector space, as we'll be using Einstein summation conventions to sum over repeated indices. The $(1/2, 1/2)$ is the vector representation carrying the usual μ, ν indices, and $(1, 0)$ and $(0, 1)$ are the imaginary-self-dual and imaginary-anti-self-dual two-forms.

Supersymmetry is a symmetry transformation of the action that relates fields in different Lorentz representations, so there are a couple important tensor product decompositions to know:

$$\begin{aligned} &(1/2, 0) \otimes (1/2, 0) = (0, 0) \oplus (1, 0) : \psi \otimes \psi' \mapsto \psi\psi' \oplus \psi\sigma_{\mu\nu}\psi \ , \qquad (\sigma_{\mu\nu} = i(\star\sigma)_{\mu\nu}) \ , \\ &(0, 1/2) \otimes (1/2, 0) = (1/2, 1/2) : \bar\psi \otimes \psi' \mapsto \bar\psi\bar\sigma_\mu\psi \ . \end{aligned} \tag{1.17}$$

The σ's in these expressions are Clebsch–Gordon coefficients for the corresponding decomposition.

The supersymmetry variation that leaves the action invariant, $\delta_\xi S = 0$, takes the following form:

$$\delta_\xi \varphi = -\epsilon_{AB}\xi^A\psi^B \ , \qquad \delta_\xi A_\mu = \bar{\xi}_A\bar{\sigma}_\mu\psi^A + c.c. \ , \qquad \delta_\xi\psi^A = -\frac{i}{2}(\sigma^{\mu\nu}\xi^A)F_{\mu\nu} + \cdots \ . \tag{1.18}$$

Here, ϵ_{AB} is the $\mathrm{SU}(2)_R$ invariant tensor.

By Noether's theorem, any continuous symmetry leads to a conserved charge. The Noether charges associated with ξ^A_α are called a supercharges and denoted Q^A_α. Using the flat, $\mathbb{Z}_2$-graded Poisson bracket on phase space that follows from this action, one can compute the algebra of the conserved charges. The result is the following:

$$\{Q^A_\alpha, \bar{Q}_{\dot\beta B}\}_+ = -2i\delta^A{}_B(\sigma^\mu)_{\alpha\dot\beta}P_\mu \ , \qquad \{Q^A_\alpha, Q^B_\beta\}_+ = -2i\epsilon^{AB}\epsilon_{\alpha\beta}\bar{Z} \ . \tag{1.19}$$

Here P_μ are the Noether charges associated with translation symmetry of the theory. In particular $-P_0 = H$, the Hamiltonian of the theory. Z, meanwhile, is known as the central charge. It takes the form

$$Z = \frac{2}{g_0^2}\int_{S^2_\infty}(iF - \star F, \varphi) = \frac{4\pi i}{g_0^2}(q_{\mathrm{m}}, \varphi_\infty) - \langle q_{\mathrm{e}}, \varphi_\infty\rangle \ , \tag{1.20}$$

where the magnetic charge is the first chern class and consistent with the previous expression, $q_{\mathrm{m}} := \frac{1}{2\pi}\int_{S^2_\infty} F$, and the electric charge $q_{\mathrm{e}}^\vee := \frac{2}{g_0^2}\int_{S^2_\infty}\star F$, is the Noether charge associated with asymptotically nontrivial gauge transformations.[8]

The algebra (1.19) is part of the $\mathcal{N} = 2$ super-Poincaré algebra, which is a $\mathbb{Z}_2$-graded extension of the ordinary Poincaré algebra. We write $\mathfrak{s} = \mathfrak{s}_{\mathrm{even}} + \mathfrak{s}_{\mathrm{odd}}$, where the odd part is generated by the Q's and the even part is

$$\mathfrak{s}_{\mathrm{even}} = \mathfrak{poin}(1,3) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \oplus \mathbb{C} \ . \tag{1.21}$$

The $\mathfrak{su}(2)_R$ generates the R-symmetry we discussed earlier, while the $\mathbb{C}$ factor is associated with the central charge (which is a central element of the algebra). There is also an internal $\mathfrak{u}(1)_R$ symmetry in the classical theory, but this is anomalous in the quantum theory, so will not be of particular importance for us.

The algebra of the Q's implies a bound on the Hamiltonian. This can be most easily extracted by introducing the linear combinations:

$$\mathcal{R}^A_\alpha = \zeta^{1/2}Q^A_\alpha + \zeta^{-1/2}(\sigma^0)_{\alpha\dot\beta}\bar{Q}^{\dot\beta A} \ , \qquad \mathcal{T}^A_\alpha = \zeta^{1/2}Q^A_\alpha - \zeta^{-1/2}(\sigma^0)_{\alpha\dot\beta}\bar{Q}^{\dot\beta A} \ , \tag{1.22}$$

where ζ is a phase, $|\zeta| = 1$. Then one computes

$$\{\mathcal{R}^A_\alpha, \mathcal{R}^B_\beta\}_+ = -4i\epsilon_{\alpha\beta}\epsilon^{AB}(H + \mathrm{Re}(\zeta^{-1}Z)) \ . \tag{1.23}$$

[8] q_{e} has been defined to sit in the dual of the Cartan, $\mathfrak{t}^\vee$ because electric charges naturally define a coupling of a charged particle to a gauge field via the line-integral: $\langle q_{\mathrm{e}}, \int A\rangle$ along the worldline of the particle.

It follows from this and the reality properties of the $\mathcal{R}$'s that

$$H + \mathrm{Re}(\zeta^{-1} Z) \geq 0 \ . \tag{1.24}$$

This is called a BPS bound after Bogomolny, Prasad, and Sommerfield, and we define *BPS field configurations* (A, φ, ψ^A) as those field configurations such that the bound is saturated. Equivalently, when the bound is saturated the $\mathcal{R}$ charges must vanish, and BPS field configurations are those that are invariant under the subset of supersymmetries corresponding to the $\mathcal{R}$'s: $\delta_\xi^{(\mathcal{R})}(A, \varphi, \psi^A) = 0$.

Setting $\varphi = \zeta^{-1}(Y + iX)$, one finds that the BPS field configurations must satisfy

$$F_{ij} - \epsilon_{ijk} D^k X = 0 \ , \qquad D^i D_i Y + [X, [X, Y]] = 0 \ , \qquad F_{i0} = D_i Y \ , \tag{1.25}$$

(and the fermions must vanish). On such a configuration, the BPS bound and the central charge can be used to obtain the expression for the energy:

$$H = \frac{4\pi}{g_0^2}(q_{\mathrm{m}}, X_\infty) + \langle q_{\mathrm{e}}, Y_\infty \rangle \ . \tag{1.26}$$

But what is ζ? There are two cases to consider.

1. The first is *framed BPS field configurations*. In this case we allow the field configurations to have singularities consistent with the presence of a defect, like before. The defects can be defined to preserve half of the supersymmetry—namely the $\mathcal{R}$-type supercharges, and thus ζ is specified by the defect. For example, we say that we have an 't Hooft defect of type ζ (at $r = 0$), denoted $L_\zeta(P)$, by demanding that

$$\varphi = i\zeta^{-1}\frac{P}{2r} + \cdots \ , \qquad F = \frac{P}{2}\sin\theta \mathrm{d}\theta \mathrm{d}\phi + \cdots \ . \tag{1.27}$$

   Framed BPS field configurations are those satisfying (1.25) as well as this singularity condition at $r = 0$.

2. The second is *vanilla BPS field configurations*. This is the case when defects are not present. Then, for every choice of ζ we have a bound, and hence, the only bound that can be saturated is the strongest bound. Thus one must vary ζ to achieve the strongest bound, and this is at the value $\zeta = \zeta_{\mathrm{van}} := -Z/|Z|$.

We close with a few comments about the system of BPS equations, (1.25), which describes dyons. We observe that the first equation is just the Bogomolny equation for monopoles, and that this can be solved independently of Y and A_0, the time component of A. Then, given a (singular) monopole (A, X), and a boundary value Y_∞, the solution to the second equation for Y will be unique. Finally the last equation then determines the electric field in terms of Y, and thus in terms of the monopole solution. In particular, it follows that the electric charge, is a $\mathfrak{t}^\vee$-valued function on the moduli space: $q_{\mathrm{e}} : \underline{\overline{\mathcal{M}}} \to \mathfrak{t}^\vee$ (or $\mathcal{M} \to \mathfrak{t}^\vee$).

2 Lecture 2: Quantum SYM and Dirac Operators on $\mathcal{M}$

Recall last time we had the $\mathcal{N} = 2$ super-Poincaré algebra of Noether charges. In particular BPS field configurations preserve the $\mathcal{R}$-type supercharges and saturate the Bogomolny bound on the Hamiltonian. The relevant part of the algebra is

$$\{\mathcal{R}_{\alpha}^{A}, \mathcal{R}_{\beta}^{B}\}_{+} = -4i\epsilon^{AB}\epsilon_{\alpha\beta}(H + \mathrm{Re}(\zeta^{-1}Z)) \ , \tag{2.1}$$

where Z is the central charge. This was a purely classical discussion with the algebra being defined by the Poisson (or more generally, Dirac) bracket on phase space.

2.1 Quantum generalities

To pass to the quantum theory, we promote the $\mathbb{Z}_2$-graded Poisson bracket to a $\mathbb{Z}_2$-graded (anti)-commutator, $\{\ ,\ \}_{\pm} \to [\ ,\ ]_{\pm} = i\{\ ,\ \}_{\pm}$, put hats on everything, and call it a day:

$$[\hat{\mathcal{R}}_{\alpha}^{A}, \hat{\mathcal{R}}_{\beta}^{B}]_{+} = 4\epsilon^{AB}\epsilon_{\alpha\beta}(\hat{H} + \mathrm{Re}(\zeta^{-1}\hat{Z})) \ . \tag{2.2}$$

More seriously, in the quantum theory we have a Hilbert space of states, and conserved charges in the classical theory are promoted to time-independent operators acting on this Hilbert space. The Hilbert space forms a unitary representation of the symmetry algebra generated by these operators. Hence we can gain some general understanding of the structure of the space of states simply by studying the representation theory of the superalgebra. The discussion must be divided according to whether or not defects are present, as the superalgebra in question depends on this.

Let's start with the standard case without defects. Then we have the full $\mathcal{N} = 2$ superalgebra, where the odd elements can be taken as the $\hat{\mathcal{R}}$'s and the $\hat{\mathcal{T}}$'s. As the $\mathcal{N} = 2$ algebra contains the usual Poincaré algebra, we first have the usual Fock space construction of multi-particle states built on tensor products of states in the one-particle Hilbert subspace.[9] The one-particle Hilbert subspace can be decomposed into unitary representations of supersymmetry following the Wigner little group approach.[10] First we Lorentz boost to the rest frame of the particle, where the eigenvalue of the Hamiltonian is the mass. Then the algebra of the $\hat{\mathcal{R}}$'s and $\hat{\mathcal{T}}$'s restricted to a mass eigenspace is

$$\begin{aligned}
&[\hat{\mathcal{R}}_{\alpha}^{A}, \hat{\mathcal{R}}_{\beta}^{B}]_{+} = 4\epsilon^{AB}\epsilon_{\alpha\beta}(M - |Z|) \ , \\
&[\hat{\mathcal{T}}_{\alpha}^{A}, \hat{\mathcal{T}}_{\beta}^{B}]_{+} = 4\epsilon^{AB}\epsilon_{\alpha\beta}(M + |Z|) \ , \\
&[\hat{\mathcal{R}}_{\alpha}^{A}, \hat{\mathcal{T}}_{\beta}^{B}]_{+} = 0 \ .
\end{aligned} \tag{2.3}$$

where Z is the eigenvalue of the central charge, and we've set $\zeta = \zeta_{\mathrm{van}} = -Z/|Z|$. The $\mathcal{R}$'s and $\mathcal{T}$'s thus form two commuting Clifford algebras and representations are constructed by acting with lowering operators on a highest weight Clifford vacuum. Then there are two

[9] See *e.g.* Streater-Wightman or Duncan.

[10] For further details see Greg Moore's PiTP lectures on BPS states and wall-crossing.

types of representations. Long representations for which $M > |Z|$ and short representations for which $M = |Z|$. In the latter case the $\mathcal{R}$'s must be represented by zero. States in short representations are called *BPS states*. We denote the subspace of BPS states as $\mathcal{H}^{\mathrm{BPS}} \subset \mathcal{H}^{\text{1-part}} \subset \mathcal{H}$.

The presence of a defect breaks the algebra of conserved charges down to the sub-superalgebra generated by the $\mathcal{R}$-type supercharges (with ζ given by the specification of the defect). Hence the Hilbert space of states is modified by the presence of the defect and is denoted $\overline{\mathcal{H}}_{L_\zeta}$ when line defects of type ζ are present. We can again diagonalize the Hamiltonian and central charge, and then look at the Wigner little group. Long representations have mass strictly greater than the bound, $M > -\operatorname{Re}(\zeta^{-1}Z)$ while short representations saturate the bound, such that the $\mathcal{R}$'s are represented by zero. *Framed BPS states* are defined as states in these short representations[11], and the subspace of framed BPS states is denoted $\overline{\mathcal{H}}^{\mathrm{BPS}}_{L_\zeta} \subset \overline{\mathcal{H}}_{L_\zeta}$.

In order to gain a more detailed description of BPS states, one must go beyond these kinematical considerations and consider the dynamics of the theory. There are two time-honored approximation schemes. One is the low-energy effective (Wilsonian) approach. When augmented by supersymmetry and other considerations, Seiberg and Witten showed that this leads to exact results for *e.g.* the possible spectrum of BPS states. The other approach is the weak-coupling or semiclassical expansion, taking the classical monopole configurations as starting point. This leads to the geometric description of BPS states we're after. We'll want to compare results from both approaches so we start with a brief synopsis of the low-energy/Seiberg–Witten analysis.

2.2 Seiberg–Witten approach

The first point is that the quantum theory has a space of vacua called the Coulomb branch, and denoted $\mathcal{B}$. In the classical theory the space of vacua can be labeled by the gauge-inequivalent asymptotic values of φ that minimize the potential. This space persists in the quantum theory because supersymmery forbids quantum corrections from lifting it. In the quantum theory we parameterize the space by complex coordinates u that can be taken as the expectation values of the gauge-invariant casimirs associated with the complex Higgs: $\{u^s\}_{s=1}^{\operatorname{rnk}\mathfrak{g}}$, with $u^s = \langle \operatorname{tr}_{\mathrm{adj}}(\varphi^s)\rangle$.

In the low-energy effective approach one identifies the massless and massive degrees of freedom over a given u, and 'integrates out' the massive ones. The masses originate from the adjoint action of the Higgs field. Over a generic vacuum the Higgs vev determines a root decomposition of the complexified Lie algebra

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} E_\alpha \cdot \mathbb{C} \ , \tag{2.4}$$

[11] The terminology of 'framed' originates from a quiver construction for such states in which the presence of line defects leads to framed quivers. It has nothing to do with the notation of 'framing' the asymptotic monopole data.

where Δ is the set of nonzero roots and E_α the corresponding raising or lowering operator. Denoting the collection of adjoint-valued fields by $\mathcal{A} = (\varphi, A, \psi^A)$, we can correspondingly expand in components $\{\mathcal{A}^I, \mathcal{A}^{(\alpha)}\}$ along the Cartan and the root directions respectively. The $\mathcal{A}^I$ are the massless d.o.f.'s while the $\mathcal{A}^{(\alpha)}$ acquire masses in terms of u.

Schematically, the low-energy effective action $S_{\text{eff}} = S_{\text{eff}}[\mathcal{A}^I]$ is defined through the path integral

$$\int [\mathcal{D}\mathcal{A}] e^{iS[\mathcal{A}]} = \int [\mathcal{D}\mathcal{A}^I] e^{iS_{\text{eff}}[\mathcal{A}^I]} \ , \tag{2.5}$$

Here $[\mathcal{D}\mathcal{A}]$ denotes a measure on the space of gauge-inequivalent field configurations and $S[\mathcal{A}]$ is the $\mathcal{N} = 2$ action. On the right side we are to have carried out the path integral over the massive modes along the root directions. This integral can be regulated and computed perturbatively in the coupling g_0 using standard techniques in QFT. At each order in the small coupling expansion the result can be expressed in terms of a derivative expansion. Typically, one is restricted to perturbative results in both of these expansions, however in this case supersymmetry strongly constrains the form of the two-derivative effective action, (which is the leading set of terms in the derivative expansion). Letting $\mathcal{A}^I = (a^I, A^I, \psi^{AI})$, where a^I are the Cartan components of the complex Higgs field, $\mathcal{N} = 2$ supersymmetry in fact determines the two-derivative effective action up to a single meromorphic functional $\mathcal{F} = \mathcal{F}[a^I; g_0]$ known as the prepotential. The weak-coupling expansion of the two-derivative action is then controlled by the weak-coupling expansion of this prepotential.

In a remarkable feat, Seiberg and Witten were able to determine the prepotential for a class of $\mathcal{N} = 2$ theories exactly, using various consistency requirements. The solution gives the functions $a^I(u)$ and $a_{\text{D},I}(u)$ in terms of period integrals of an auxiliary Riemann surface. This implicitly gives a relation between a_{D} and a, and the prepotential is defined by $a_{\text{D},I} = \partial\mathcal{F}/\partial a^I$. Solutions have since been obtained for many more theories, including the whole class of theories we consider here.

One can inquire about the form of the $\mathcal{N} = 2$ algebra in this language. The spectrum of the central charge operator can in fact be computed exactly. It is labeled by electromagnetic charges:

$$\gamma_{\text{m}}^I := \frac{1}{2\pi} \int_{S^2_\infty} F^I \ , \qquad \gamma_{\text{e},I} = \tau_{IJ} \int_{S^2_\infty} \star F^J \ , \qquad \text{where} \quad \tau_{IJ} := \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} \ , \tag{2.6}$$

where $F^I = \mathrm{d}A^I$, and the eigenvalues are

$$Z_\gamma(u) = a_{\text{D},I}(u)\gamma_{\text{m}}^I + a^I(u)\gamma_{\text{e},I} \ . \tag{2.7}$$

The result is exact because the form of the higher derivative terms in the effective action, which are not incorporated by the SW solution, is constrained by supersymmetry in such a way that they cannot contribute to the central charge. Then by the Bogomolny bound this gives the mass of BPS states when such states are present in the spectrum.

Before describing this further we introduce a slightly more invariant language. The $a^I(u)$ should be viewed as local coordinates on $\mathcal{B}$; recall they were defined in terms of the root decomposition specified by φ_∞. They do not extend over all of $\mathcal{B}$. Globally, supersymmetry dictates that $\mathcal{B}$ has a certain type of geometry known as special Kähler. The electromagnetic charge is a section of a local system $\Gamma \to \mathcal{B}$ over $\mathcal{B}$. This means we have a fibration of charge lattices over $\mathcal{B}$ that is locally trivial but globally can undergo monodromy. So the picture is the following... The lattice is equipped with a symplectic pairing $\langle\!\langle\ ,\ \rangle\!\rangle$ which is invariant under these transformations. (The monodromy transformations are electromagnetic duality transformations.) The $a^I(u)$ are systems of special coordinates on $\mathcal{B}$ that give a local trivialization of the lattice into magnetic and electric components. In the trivialization described above we can identify these components with

$$\Gamma_u \cong \Lambda_{\rm mw} \oplus \Lambda_{\rm rt} \cong \mathfrak{t} \oplus \mathfrak{t}^\vee \ , \tag{2.8}$$

and the symplectic pairing is given in terms of the pairing between $\mathfrak{t}^\vee$ and $\mathfrak{t}$:

$$\langle\!\langle \gamma_1, \gamma_2 \rangle\!\rangle = \langle \gamma_{2,\rm e}, \gamma_{1,\rm m} \rangle - \langle \gamma_{1,\rm e}, \gamma_{2,\rm m} \rangle \ . \tag{2.9}$$

Then the Hilbert subspace of vanilla BPS states is fibered over $\mathcal{B}$, and can be graded by electromagnetic charges. Furthermore, the action of the $\mathcal{T}$ supersymmetries produces a four-dimensional representation of the Clifford algebra tensored with the representation of the Clifford vacuum. The former is called the half-hypermultiplet, it is associated with the center of mass degrees of freedom of the one-particle state. The latter is associated with the internal degrees of freedom and, based on the general principles discussed earlier, it is a representation space for the algebra of the Wigner little group, $\mathfrak{so}(3) \oplus \mathfrak{su}(2)_R$.

The mass of these states is $|Z_\gamma(u)|$. The linearity of $Z_\gamma(u)$ with respect to the charge (along with conservation of charge) implies, by the triangle inequality, that BPS states are stable as we vary u, except at *marginal stability walls*, where the central charges associated to two potential constituents align. These are co-dimension one walls defined by the conditions:

$$W(\gamma_1, \gamma_2) := \left\{ u \in \mathcal{B} \ \middle| \ \overline{Z_{\gamma_1}(u)} Z_{\gamma_2}(u) \in \mathbb{R}_+ \ , \ \& \ \begin{array}{r} \gamma_1 + \gamma_2 \ , = \gamma \\ \langle\!\langle \gamma_1, \gamma_2 \rangle\!\rangle \neq 0, \\ (\mathcal{H}_0^{\rm BPS})_{u,\gamma_{1,2}} \neq 0 \end{array} \right\} \ . \tag{2.10}$$

Upon crossing such a wall, states can decay into two constituents (or two groups of constituents) with charges γ_1 and γ_2. Note the last two conditions ensure that the bound state could have existed in the first place. The nonzero pairing is required in order for the constituents to bind, while the last condition ensures that the constituents are available in the spectrum. As one approaches the wall $W(\gamma_1, \gamma_2)$, we have a point dyon picture in the low energy effective theory that describes the situation. The two constituents are bound together, but as the wall is approached, the binding strength becomes weaker and weaker, until the bound state radius goes to infinity and this ceases to be a single-particle state.

Similarly the Hilbert subspace of framed BPS states is fibered over $\mathcal{B}$ and can be graded by electromagnetic charges. Likewise, the little group analysis allows these spaces to be arbitrary $\mathfrak{so}(3) \oplus \mathfrak{su}(2)_R$ representations. The mass of framed BPS states is given by $-\operatorname{Re}\left[\zeta^{-1} Z_\gamma(u)\right]$. Theses states are also generically stable, but there are co-dimension one marginal stability walls. They are defined by a single 'halo' charge and given as

$$W(\gamma_{\rm h}) = \left\{ (u,\zeta) \mid \zeta^{-1} Z_{\gamma_{\rm h}}(u) \in \mathbb{R}_- \ , \ (\mathcal{H}_0^{\rm BPS})_{u,\gamma_{\rm h}} \neq 0 \right\} \ . \tag{2.11}$$

In this case we have the core halo picture in which the framed BPS state is viewed as the bound state of a bunch of vanilla particles (making up the halo) to an infinitely massive core particle (the low-energy effective description of the line defect). As we approach the wall, the halo particles become less and less bound.

Finally, their are explicit formulae for how indices associated with these spaces change when a wall is crossed. The indices are defined as traces over the appropriate Hilbert space, of $(-1)^F$ where F is the fermion number. This can be generalized to virtual characters that keep track of spin information as well. Then, for example, the vanilla WCF is given by the Kontesvich–Soibelman formula, but we will not need these details for this talk.

2.3 The semiclassical approach

Let us turn now to the semiclassical approach. The starting point is the classical BPS field configurations we had from last time:

$$F_{ij} - \epsilon_{ijk} D^k X = 0 \ , \qquad D^i D_i Y + [X,[X,Y]] = 0 \ , \qquad F_{i0} = D_i Y \ , \tag{2.12}$$

The idea here is to quantize fluctuations around the monopole and work perturbatively in g_0. Schematically, we write

$$\mathcal{A}(t,\vec{x}) = \mathcal{A}_{\rm mono}(\vec{x}; Z^m(t)) + g_0 \delta\mathcal{A} \ . \tag{2.13}$$

Here the Z^m are known as collective coordinates, and contain both bosonic and fermionic degrees of freedom:

$$Z^m(t) = (z^m(t), \eta^m(t)) \tag{2.14}$$

where $m,n, = 1,\ldots,\dim\underline{\overline{\mathcal{M}}}$. The $\{z^m\}$ are local coordinates on the moduli space—these are the original collective coordinates of Manton—while the $\{\eta^m\}$ parameterize the fibers of a certain index bundle over the moduli space, which turns out to be isomorphic to the tangent bundle. They are related to the z^m by supersymmetry.

Meanwhile the $\delta\mathcal{A}$ term represents field fluctuations parameterizing the directions orthogonal to the moduli space in the full field configuration space. One uses a saddle point approximation to carry out the path integral over these degrees of freedom, resulting in a supersymmetric quantum mechanics for the collective coordinates. The supersymmetries of the quantum mechanics descend directly from the preserved $\mathcal{R}$-supersymmetries of the background.

The canonical commutation relations for the η^m, $[\eta^m, \eta^n]_+ = 2(g_{\overline{\mathcal{M}}})^{mn}$ give a Clifford algebra, which, on a hyperkähler manifold, can be represented in one of two equivalent ways. We can either view states in the quantum mechanics as sections of the Dirac spinor bundle on $\overline{\mathcal{M}}$ (or $\mathcal{M}$), or, if we choose a complex structure, they can be viewed as $(0,*)$-forms.

We are not interested in general states, however, but BPS states. These are states that are annihilated by the supercharges. In the quantization via spinors, for example, one[12] of the supercharges turns out to be represented as a Dirac-like operator, $\hat{Q} = i\not{D}^{\mathcal{Y}}$. However this is not the ordinary Dirac operator, but rather a modification of the ordinary Dirac operator where we subtract a term involving Clifford contraction with a certain vector field. The vector field is a tri-holomoprhic Killing associated to the vev of the secondary Higgs field. Recall the map G from the Cartan subalgebra into the space of triholomorphic Killing vectors. Then, at leading order in the semiclassical expansion, the $\mathcal{X}, \mathcal{Y}$ appearing here are given by $\mathcal{X} = X_\infty$ and $\mathcal{Y} = \frac{4\pi}{g_0^2} Y_\infty$.

We will also need the electric charge operator. Recall that, classically, it was a function on the moduli space. In the quantum theory it is promoted to..., where $\mathcal{L}_{\mathrm{G}(h^I)}$ is the Kossman–Lie derivative on spinors along a basis of triholomorphic Killing vectors. Since these are properly normalized to generate periodic isometries, the eigenvalues of the electric charge operator will take values in the root lattice. One can show that the supercharge and electric charge operator commute, and thus the kernel of $\hat{Q}$ can be decomposed into eigenspaces of the electric charge operator.

Hence we define the semiclassical space of framed as follows.

$$\overline{\mathcal{H}}_{P,\mathcal{X},\mathcal{Y},q_{\mathrm{m}},q_{\mathrm{e}}} := \ker^{(q_{\mathrm{e}})}_{L^2} \left(\not{D}^{\mathcal{Y}}_{\overline{\mathcal{M}}(P;q_m,\mathcal{X})} \right) . \tag{2.15}$$

The vanilla analog requires a bit of extra work, due to the center of mass factor. Recall that the universal cover of the moduli space is $\widetilde{\mathcal{M}} = \mathbb{R}^4 \times \mathcal{M}_0$. There are no L^2 normalizable zero-modes of the Dirac-like operator on this space due to the center of mass factor. This is physically expected however. We can decompose the spinor bundle on the total space into a spinor bundle on each factor, and we expect that the spinors on the $\mathbb{R}^4$ should be plane-wave normalizable, as they represent one-particle states. What we demand is L^2-normalizability on the strongly centered space, along with an appropriate equivariance such that they give well-defined spinors on the quotient $\mathcal{M}$.

Without going into the details, let me just state the result. Decompose $\mathcal{Y}$ as $\mathcal{Y} = \mathcal{Y}^{\mathrm{cm}} + \mathcal{Y}^0$, where $\mathcal{Y}^{\mathrm{cm}}$ is the projection of $\mathcal{Y}$ along $\mathcal{X}$ and $\mathcal{Y}^0$ is perpendicular to q_m with respect to the Killing form. The latter might sound strange, but the reason we do this is because of the rather peculiar identity $g_{\mathcal{M}}(\mathrm{G}(\mathcal{X}), \mathrm{G}(\mathcal{Y})) = (q_{\mathrm{m}}, \mathcal{Y})$.[13] Hence if $\mathcal{Y}^0$ is Killing

[12]It follows from the supersymmetry algebra that if one of the supercharges annihilates a state then all of them do, so it is sufficient to consider just this one.

[13]This can be demonstrated from the definition of G in terms of the action on (A, X) by gauge transformations, the definition of the metric in terms of integration over $\mathbb{R}^3$ of bosonic zero modes, and integration by parts.

orthogonal to $q_{\rm m}$, then $\mathrm{G}(\mathcal{Y}^0)$ will be metric orthogonal to $\mathrm{G}(\mathcal{X})$ and will hence restrict to a Killing vector on $\mathcal{M}_0$. Note this decomposition of $\mathcal{Y}$ is always possible because (non-empty) monopole moduli spaces will never have an $\mathcal{X}$ and $q_{\rm m}$ that are Killing orthogonal—that would correspond to a zero energy monopole.[14]

Dually, decompose the electric charge as $q_{\rm e} = q_{\rm e}^{\rm cm} + q_{\rm e}^0$, where $q_{\rm e}^{\rm cm}$ is the component of the electric charge parallel to the dual of the magnetic charge, and $q_{\rm e}^0$ annihilates $\mathcal{X}$: $\langle q_{\rm e}^0, \mathcal{X}\rangle = 0$. Then the centered semiclassical vanilla BPS space is

$$(\mathcal{H}_0^{\rm scBPS})_{\mathcal{X},\mathcal{Y}^0,q_m,q_e} := \left[\ker_{L^2}^{(q_{\rm e}^0)}\left(\not{D}^{\mathcal{Y}^0}_{\mathcal{M}_0(q_{\rm m},\mathcal{X})}\right)\right]^{\mathbb{Z}_k} , \tag{2.16}$$

where k is again the gcd of the components of the dual of the magnetic charge along the simple roots. What's happening with the equivariance condition is that quantization of electric charge ensures that states[15] descend to the quotient of of the universal cover by the $k\mathbb{Z}$ subgroup of the group of deck transformations associated to gauge transformations (the hyperholomorphic T-action). However, an additional $k\mathbb{Z}/\mathbb{Z}$ equivariance condition must be imposed to make sure they descend to well-defined states on $\mathcal{M}$. The equivariance condition specifies $q_{\rm e}^{\rm cm}$ up to integer shifts by $q_{\rm m}^\vee$. Hence, for each state in (2.16) we have an infinite tower of dyons corresponding to $q_{\rm e} \to q_{\rm e} + n q_{\rm m}^\vee$, $n \in \mathbb{Z}$. We call this the *Julia–Zee tower* since it generalizes the tower of standard $\mathfrak{su}(2)$ dyons with magnetic charge 1, first discovered by Julia and Zee, and reduces to this in that special case.[16]

In the above semiclassical descriptions of BPS states we used the spinor language. If we use the isomorphism with $(0,*)$-forms, then we would instead talk about the L^2 cohomology of a Dolbeault-like operator. Specifically:

$$\text{for framed:} \qquad \bar\partial - i\mathrm{G}(\mathcal{Y})^{(0,1)} \wedge \qquad \text{on} \qquad L^2(\Lambda^* T^{(0,1)}(\underline{\overline{\mathcal{M}}})) , \tag{2.17}$$

$$\text{for vanilla:} \qquad \bar\partial - i\mathrm{G}(\mathcal{Y}^0)^{(0,1)} \wedge \qquad \text{on} \qquad L^2(\Lambda^* T^{(0,1)}(\mathcal{M}_0)) , \tag{2.18}$$

where $\mathrm{G}(h)^{(0,1)}$ is the anti-holomorphic part of the one-form obtained by dualizing $\mathrm{G}(h)$ with respect to the metric. (One can again restrict the cohomology to electric charge eigenspaces, and in the latter case to the subspace satisfying the appropriate equivariance condition.)

2.4 The Seiberg–Witten ↔ semiclassical map

We've now given two descriptions of the space of (framed) BPS states, one in the Seiberg–Witten picture, and one in the semiclassical picture. On a regime of overlapping validity

[14] And here we are assuming that all components of the magnetic charge are strictly positive. If not, then one should view the monopole as an embedded monopole from a smaller gauge group where the magnetic charge does have this property.

[15] that is, spinors in the above kernel tensored with spinors in the (non-L^2) kernel of the $\mathbb{R}^4$ part of the Dirac operator

[16] Meanwhile, $\mathcal{Y}^{\rm cm}$ is actually fixed by the other asymptotic data and is not an independent quantity. This follows directly from the BPS equations. Specifically, $(q_{\rm m}, \mathcal{Y}^{\rm cm}) + \langle q_{\rm e}, \mathcal{X}\rangle = 0$.

they must agree. The real question is, how are the parameters that go into the two descriptions related? We conjecture the following for framed BPS states,

$$\overline{\underline{\mathcal{H}}}^{\rm scBPS}_{P,\mathcal{X},\mathcal{Y},q_{\rm m},q_{\rm e}} \cong \overline{\underline{\mathcal{H}}}^{\rm BPS}_{L_\zeta(P),u,\gamma} \tag{2.19}$$

provided

$$\mathcal{X} = {\rm Im}(\zeta^{-1}a(u)) \ , \qquad \mathcal{Y} = {\rm Im}(\zeta^{-1}a_{\rm D}(u)) \ , \qquad q_{\rm m} \oplus q_{\rm e} = \gamma \ . \tag{2.20}$$

For vanilla BPS states we conjecture

$$(\mathcal{H}_0^{\rm scBPS})_{\mathcal{X},\mathcal{Y}^0,q_m,q_e} \cong (\mathcal{H}_0^{\rm BPS})_{u,\gamma} \ , \tag{2.21}$$

provided the same relations hold with $\zeta \to \zeta_{\rm van} := -Z_\gamma(u)/|Z_\gamma(u)|$. Here we've defined Cartan-valued $a, a_{\rm D}$ by taking $a(u) = a^I(u)H_I$ and $a_{\rm D} = \sum_I a_{{\rm D},I}(\alpha_I)^\vee$.

Some fine print that must be included is the following. A choice of special coordinates $(a^I, a_{{\rm D},J})$ and corresponding splitting of the electromagnetic charge lattices $\Gamma = \Gamma_{\rm m} \oplus \Gamma_{\rm e}$ requires the specification of a 'duality frame.' In the above the frame is determined by the condition that $\mathcal{X}$ be in the fundamental Weyl chamber. Secondly, the equivalence is only expected to hold over an appropriately defined weak-coupling regime of the Coulomb branch. One reasonable definition of the regime $\mathcal{B}_{\rm wc} \subset \mathcal{B}$ would be in terms of the radius of convergence of the weak-coupling expansion of the Seiberg–Witten prepotential. This is motivated by the observation that this is the regime where one could, *in principle*, sum all saddle-point contributions to the path integral over the fluctuations $\delta\mathcal{A}$ in (2.13). If one could do this, then the result for the BPS spectrum defined from the quantum mechanics must match the Seiberg–Witten result. This regime excludes strong-coupling chambers that are the interior of compact sets in $\mathcal{B}'$, and keeps us away from the complex codimension one singularities associated with (classical) symmetry enhancement of the gauge group.

What is the motivation for the form of the conjecture, (2.20)? First, these definitions reduce to the leading order semiclassical results we mentioned previously, $\mathcal{X} \to X_\infty$, $\mathcal{Y} \to \frac{4\pi}{g_0^2} Y_\infty$, as $a \to \varphi_\infty = \zeta^{-1}(Y_\infty + iX_\infty)$ in this limit. Second, they pass a limited check at one-loop that we performed by explicitly computing the one-loop correction to the soliton mass in the semiclassical framework. Third, and by far the strongest evidence, is that they give the right identification in a specific (and highly nontrivial) example where both descriptions of the BPS Hilbert subspaces are computable in detail. We will comment on this example in the next lecture.

3 Lecture III: Predictions from physics for L^2 cohomology

Let's summarize the situation from last time. We constructed families of Dirac-like operators on the singular monopole moduli spaces $\overline{\underline{\mathcal{M}}}$, and the strongly centered moduli spaces $\mathcal{M}_0$:

$$\not{D}^{\mathcal{Y}}_{\overline{\underline{\mathcal{M}}}(P;q_{\rm m},\mathcal{X})} \equiv \not{D}^{\mathcal{Y}} \ , \qquad \not{D}^{\mathcal{Y}^0}_{\mathcal{M}_0(q_{\rm m},\mathcal{X})} \equiv \not{D}^{\mathcal{Y}^0} \ . \tag{3.1}$$

These depend on $\mathcal{X} \in W_+ \subset \mathfrak{t}$, the fundamental Weyl chamber, and $\mathcal{Y} \in \mathfrak{t}$, or $\mathcal{Y}^0 \in \mathfrak{t}_{q_{\mathrm{m}}}^{\perp}$, the subspace of the Cartan that is Killing orthogonal to the magnetic charge. Equivalently we have Dolbeault-like operators $\bar{\partial}^{\mathcal{Y}}$ and $\bar{\partial}^{\mathcal{Y}^0}$.

The isometries of $\overline{\mathcal{M}}$ and $\mathcal{M}_0$ are generated by Killing vectors forming an $\mathfrak{so}(3) \oplus \mathfrak{t}$ and $\mathfrak{so}(3) \oplus \mathfrak{t}_{q_{\mathrm{m}}}^{\perp}$ algebra. These actions can be lifted to actions on the Dirac spinor bundle. In particular q_{e} and q_{e}^0 are characters for the torus groups associated with the last factors, and the L^2-kernel of the Dirac operators can be graded by these characters. We identified these kernels with the spaces of framed and vanilla BPS states:

$$\ker_{L^2}^{(q_{\mathrm{e}}^0)} \not{D}^{\mathcal{Y}} \cong \overline{\underline{\mathcal{H}}}_{L_\zeta,u,\gamma}^{\mathrm{BPS}} , \qquad \left[\ker_{L^2}^{(q_{\mathrm{e}}^0)} \not{D}^{\mathcal{Y}^0}\right]^{\mathbb{Z}_k} \cong (\mathcal{H}_0^{\mathrm{BPS}})_{u,\gamma} . \tag{3.2}$$

Here the $k\cdot\mathbb{Z}/\mathbb{Z}$ equivariance condition, together with q_{e}^0, determine the electric charge q_{e} up to integer shifts by $q_{\mathrm{m}}^{\vee}$. Hence the BPS spaces for this Julia–Zee tower of electromagnetic charges are the same. The identifications of these spaces are made subject to the map described under (2.20).

Now we turn to some consequences of the identifications (3.2).

3.1 P1: No exotics as a generalized Sen conjecture

What is no-exotics? The Wigner little group analysis permits $\overline{\underline{\mathcal{H}}}_{L_\zeta,u,\gamma}^{\mathrm{BPS}}$, $(\mathcal{H}_0^{\mathrm{BPS}})_{u,\gamma}$, to be arbitrary $\mathrm{SU}(2)_R$ representations. But all 'experimental evidence' to date (and there is quite a lot) shows that they always transform trivially. The no-exotics conjecture is the claim that this is always true on the Coulomb branch of any $\mathcal{N} = 2$ theory. 'Exotic' states in this context are states that transform nontrivially under $\mathrm{SU}(2)_R$. Since this conjecture was made in 2010 by Gaiotto–Moore–Neitzke [], it has been proven for the type of theories we are considering here (theories without matter hypermultiplets), when the gauge group is simply laced []. The proof utilizes yet a different representation of BPS states in terms of curves in Calabi–Yau three-folds. This representation of BPS states follows from the 'geometric engineering' of $\mathcal{N} = 2$ supersymmetric gauge theories in string theory.[17] A more generally applicable argument based on the analytic structure of matrix elements of the R-symmetry (Noether) current operators between BPS states has also been given by Cordova and Dumitrescu. So let's take no-exotics as a well-motivated hypothesis based on $\mathcal{N} = 2$ supersymmetric quantum Yang–Mills.

We then must understand what it means semiclassically, and in order to do that we must first understand the realization of $\mathrm{SU}(2)_R$ in the semiclassical language. For this it is convenient to use the forms representation of states, where states are L^2 sums of $(0, q)$ forms, $\Psi = \oplus_q \Psi^{(0,q)}$.[18] On our moduli spaces we have a triplet of Kähler forms. Take the

[17] One can determine the geometric meaning of $SU(2)_R$ in this context and turn the statement of no-exotics into an explicit statement about curves in CY3's.

[18] There is also a nice characterization in the spinor language: There is an $\mathrm{SU}(2)_R$ action via endomorphisms of the tangent space that form the commutant of the holonomy group $\mathrm{Sp}(N) \subset \mathrm{SO}(4N)$, where $4N = \dim \overline{\mathcal{M}}$ or $\dim \mathcal{M}_0$. This action can be lifted to the Dirac spinor bundle.

third one to be 'the' Kahler form, defining our complex coordinates and use the other two to construct a holomorphic-symplectic form and its conjugate. Then, the action of $\mathrm{SU}(2)_R$ on general states is given by the following. The Cartan generator on $\lambda^{(q)}$ gives $\frac{1}{2}(q-N)$ times $\lambda^{(q)}$, where we take the dimension of $\overline{\mathcal{M}}$ or $\mathcal{M}_0$ to be $4N$. The raising operator acts by wedging with ω^+ and the lowering operator by contracting with ω^-. These operations in fact define an $\mathfrak{sl}(2)$ Lefchetz action on the anti-holomorphic tangent bundle which descends to the cohomology of the twisted Dolbeault operator defining the BPS states. Hence no exotics holds if and only if

P1: all nontrivial L^2 cohomology of our supercharge operators $\bar{\partial}^{\mathcal{Y}}$ and $\bar{\partial}^{\mathcal{Y}^0}$, is in the middle degree (by the vanishing of the $\hat{\mathcal{I}}^3$ action)—*i.e.* $(0,N)$-forms—and is furthermore Lefshetz-primitive.

This is a strong statement about the cohomology of a large class of differential operators on a large class of hyperkähler manifolds. It is reminiscent of Sen's famous conjecture concerning the existence of harmonic forms on monopole moduli space, which he derived from considerations of the 4D $\mathcal{N}=4$ theory and semiclassical analysis.[19] In fact, this result can be viewed as a generalization of the original Sen conjecture, but to fit Sen's result within this framework we have to extend our analysis to cover $\mathcal{N}=2$ theories with matter hypermultiplets. The reason is that the $\mathcal{N}=4$ theory that Sen analyzed is a special case of an $\mathcal{N}=2$ theory with matter—specifically an adjoint-valued matter hypermultiplet.[20]

This was recently demonstrated by Brennan and Moore []. It has been known since work of [] that matter fermions carry zero-modes in the monopole background which are associated with an index bundle of a Dirac operator. Quantization of these additional collective coordinates amounts to twisting the Dirac operator by connection on the Spin bundle associated to the matter index bundle. (The Spin bundle arises from representing the Clifford algebra that the additional collective coordinates satisfy.) The matter bundle comes with a hyperholomorphic connection given in terms of Atiyah and Singers 'universal connection.' In the special case where the matter transforms in the adjoint representation, the matter bundle is isomorphic to the holomorphic tangent bundle, and hence the spin bundle gives us a copy of $\Lambda^* T^{(1,0)}$. Hence states in the quantum mechanics are represented by $\Lambda^* T^{(1,0)}$ bundle-valued $(0,*)$ L^2 forms—*i.e.* by general L^2 forms. The L^2-kernel of the new twisted Dirac operator is identified with the L^2 de-Rham cohomology, and no-exotics still leads to the statement that the cohomology vanishes outside the middle degree. (See Brennan–Moore for more details [].)

[19] It is also reminiscent of the Vafa–Witten conjecture reviewed in Richard Melrose's talk, and results discussed in Francesco Bei's talk.

[20] Here 'hypermultiplet' refers to the supersymmetry representation content; in this context it involves two additional complex Higgs fields and two additional Weyl fermions.

3.2 P2: Wall-crossing for the Dirac kernel

The second prediction is concerned with translating the stability and wall-crossing properties into statements about where the Dirac-like operator fail to be Fredholm and how their kernels jump. Something I didn't have time to explain is that one consequence of no-exotics in spinor language is that the kernel of the Dirac operator must be chiral.[21] Therefore the index computes the dimension of the kernel.

Then, applying the map (2.20) to the formulae from earlier for the vanilla walls, we conclude the following:

P2a: Dirac operators in the family $\not{D}^{\mathcal{Y}_0}_{\mathcal{M}_0(q_{\rm m},\mathcal{X})}$ for $(\mathcal{X},\mathcal{Y}^0)\in W_+\times \mathfrak{t}^{\perp}_{q_{\rm m}}$ fail to be Fredholm only if there are charges $q_{1,2}=q_{1,2,\rm m}\oplus q_{1,2,\rm e}\in \Lambda_{\rm cr}\oplus\Lambda_{\rm rt}$ such that

- $q_{1,\rm m}+q_{2,\rm m}=q_{\rm m}$,
- $\langle\!\langle q_1,q_2\rangle\!\rangle\neq 0$,
- $\ker_{L^2}(\not{D}^{\mathcal{Y}_0}_{\mathcal{M}_0(q_{1,\rm m},\mathcal{X})})\neq 0$, and $\ker_{L^2}(\not{D}^{\mathcal{Y}_0}_{\mathcal{M}_0(q_{1,\rm m},\mathcal{X})})\neq 0$, and
- $(\mathcal{Y}^0,\gamma_{1,\rm m})+\langle\gamma_{1,\rm e}-\gamma^{\vee}_{1,\rm m}\frac{\langle\gamma_{\rm e},\mathcal{X}\rangle}{(\gamma_{\rm m},\mathcal{X})},\mathcal{X}\rangle=0$,

where $\gamma_{\rm e}=\gamma_{1,\rm e}+\gamma_{2,\rm e}$. Upon crossing the real co-dimension one wall defined by the last equation, the kernel of the Dirac operator will jump in the way determined by the (Kontsevich–Soibelman) wall-crossing formula for $\Omega(u,\gamma)$, (with u,γ given through (2.20)). Also, this last condition is the statement that $\mathrm{Im}(Z_{q_1}(u)\overline{Z_{q_2}(u)})=0$. There is also the condition from (2.10) that $\mathrm{Re}(Z_{q_1}(u)\overline{Z_{q_2}(u)})>0$, however one can show that this condition is always satisfied in the weak-coupling regime of the Coulomb branch where the semiclassical analysis applies [].

Note that although the last condition above appears to be asymmetric in q_1,q_2, it is in fact equivalent to the same equation with $q_1\to q_2$, due to the fact that $(\mathcal{Y}^0,\gamma_{1,\rm m}+\gamma_{2,\rm m})=0$.

Analogously, for the framed case we have:

P2b: For singular monopole moduli spaces, Dirac operators in the family $\not{D}^{\mathcal{Y}}_{\underline{\mathcal{M}}(P;q_m,\mathcal{X})}$ for $(\mathcal{X},\mathcal{Y})\in W_+\times\mathfrak{t}$ are Fredholm except on real co-dimension one walls defined by 'halo' charges $q_{\rm h}=q_{\rm h,m}\oplus q_{\rm h,e}\in\Lambda_{\rm cr}\oplus\Lambda_{\rm e}$ such that

- $\ker_{L^2}(\not{D}^{\mathcal{Y}^0}_{\mathcal{M}_0(q_{\rm h,m},\mathcal{X})})\neq 0$, and
- $(\mathcal{Y},\gamma_{\rm h,m})+\langle\gamma_{\rm h,e},\mathcal{X}\rangle=0$.

Upon crossing these walls, the kernel jumps according to the (Gaiotto–Moore–Neitzke) wall-crossing formula for the framed indices $\overline{\underline{\Omega}}(L_\zeta(P),u,\gamma)$.

[21] The action of the nontrivial central element, $-1\in\mathrm{SU}(2)_R$, on Dirac spinors is given by the action of the Clifford volume element. Hence if the spinor is to be invariant under this action then it restricts to the positive chirality spinor bundle.

3.3 An example

3.4 Conclusions

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