Exotic Vortices and their Dynamics

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Outline

- I. Vortices
- II. Standard Vortices
- III. Exotic Vortices
- IV. Summary

I. Vortices

- A vortex is a gauge theory soliton on a 2-d Riemann surface M. It couples a complex Higgs field ϕ (with no singularities) to a U(1) connection a. A zero of ϕ represents a vortex centre.
- On M, with $z = x_1 + ix_2$ a (local) complex coordinate, the metric is
$$ds_0^2 = \Omega_0(z, \bar{z})\, dz d\bar{z}\,.$$
The total area A_0 of M plays an important role in the theory. We specialise later to surfaces with constant curvature.
- To have N vortices with positive multiplicity, the first Chern number needs to be N. Physically, there is a magnetic flux $2\pi N$.

II. Standard Vortices

- The (Bogomolny) vortex equations are

$$\begin{aligned} D_{\bar{z}}\phi &= 0\,, \\ *f &= 1 - |\phi|^2\,. \end{aligned}$$

- Here $*f = \frac{1}{\Omega_0} f_{12} = \frac{1}{\Omega_0}(\partial_1 a_2 - \partial_2 a_1)$ is the magnetic field.
- The first equation $\partial_{\bar{z}}\phi - ia_{\bar{z}}\phi = 0$ can be solved for a:

$$a_{\bar{z}} = -i\partial_{\bar{z}}(\log\phi)\,, \quad a_z = i\partial_z(\log\overline{\phi})\,.$$

- The second equation then reduces to

$$-\frac{1}{2\Omega_0}\nabla^2 \log|\phi|^2 = 1 - |\phi|^2\,.$$

- It is convenient to set $|\phi|^2 = \phi\overline{\phi} = e^{2u}$. Then

$$-\frac{1}{\Omega_0}\nabla^2 u = 1 - e^{2u} \,,$$

  with the Beltrami Laplacian of u on the left. This is the Taubes vortex equation.
- u has logarithmic singularities at the zeros of ϕ, so there are N additional delta functions at the vortex centres.
- N-vortex solutions exist on M provided $A_0 > 2\pi N$ (Taubes, Bradlow, Garcia-Prada).
- The moduli space is $\mathcal{M}_N = M^N_{\rm symm}$, as there is a unique vortex given N unordered (possibly coincident) points on M.

Baptista Geometry of Vortices

- Vortices have a geometric interpretation. Define a new, Baptista metric on M

$$ds^2 = |\phi|^2 ds_0^2 = e^{2u} ds_0^2 \,.$$

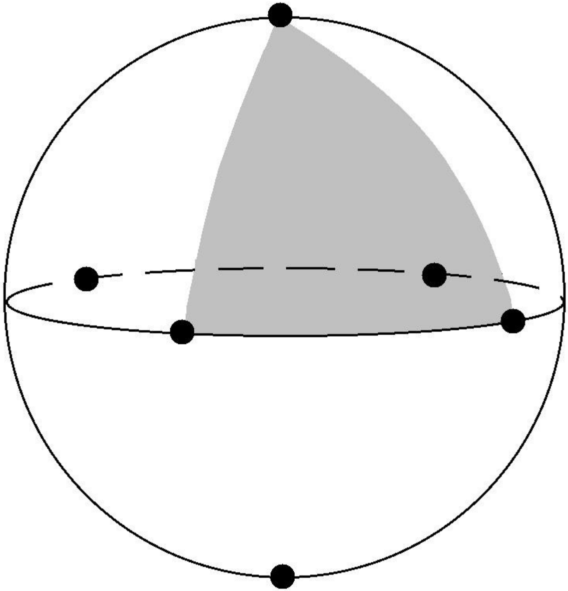
- This is conformal to the original metric, with conformal factor $\Omega = e^{2u}\Omega_0$, but has conical singularities with cone angle 4π at the N vortex centres.
- The Taubes equation can be expressed as

$$(K+1)\Omega = (K_0+1)\Omega_0 \,,$$

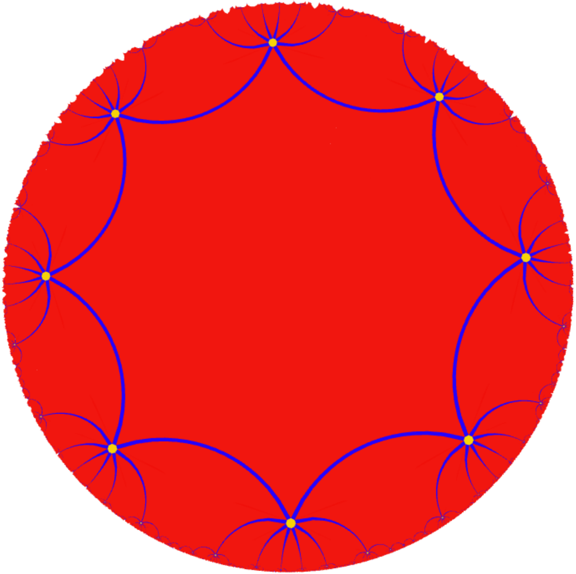
where K, K_0 are the Gaussian curvatures of Ω, Ω_0. Gauss–Bonnet, allowing for the N conical singularities, reproduces the Bradlow constraint $A_0 > 2\pi N$.

Integrable Vortices

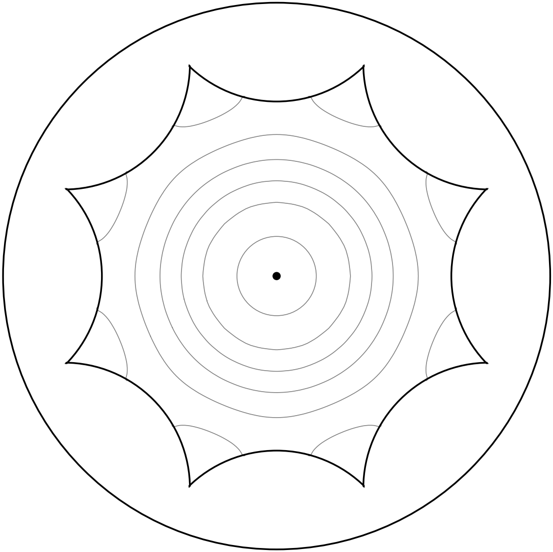
- The vortex equations are integrable if M is hyperbolic, i.e. if $K_0 = -1$. This was known to Witten – the vortex eqs. are a dimensional reduction of anti-self-dual SU(2) Yang–Mills eqs. on $\mathbb{R}^4 \sim \mathbb{H}^2 \times S^2$. Vortices on $\mathbb{H}^2$ are SO(3)-invariant instantons.
- Explicit solutions are known on $\mathbb{H}^2$ and also on the hyperbolic Bolza surface of genus 2 (Maldonado and NSM). The Bradlow constraint on a compact genus $\mathfrak{g}$ surface with $K_0 = -1$ is $N < 2\mathfrak{g} - 2$.
- Note that $K = -1$ in this case; the Baptista metric on M is hyperbolic, with conical singularities.
- The moduli space $\mathcal{M}_N$ becomes a moduli space of punctured Riemann surfaces with conical, hyperbolic metrics. One could extend $\mathcal{M}_N$ to allow for the moduli of the background Riemann surface M.



Bolza surface double covers the Riemann sphere



{8,8} tessellation of $\mathbb{H}^2$ by Bolza octagons



Contours of $|\phi|^2 = e^{2u}$ for Bolza vortex at centre.

Vortex Dynamics and Moduli Space Geometry

- Vortices satisfying the Bogomolny equations are *minima* of a U(1) Yang–Mills–Higgs energy in 2-d.
- There is a dynamical theory in 2+1 dimensions. Vortices can move, and they have kinetic energy. Restricted to $\mathcal{M}_N$, the kinetic energy is a quadratic form in vortex velocities (tangent vectors), and defines a metric on $\mathcal{M}_N$. Slowly moving vortices follow geodesics in $\mathcal{M}_N$.
- The metric on $\mathcal{M}_N$ is Kähler, and there is a formula for it based on local properties of u near each vortex centre (Strachan–Samols localization).
- What is the relation between the Strachan–Samols metric and the Weil–Petersson metric on punctured Riemann surfaces (with 4π cone angles)? We don't know.

III. Exotic Vortices

- A more general Taubes-type vortex equation is

$$-\frac{1}{\Omega_0}\nabla^2 u = -C_0 + Ce^{2u}\,.$$

  Both constants C_0 and C can be scaled to either $-1, 0, +1$, so there are nine cases.

- The LHS is the magnetic field. Its integral over M must be positive if vortex number $N > 0$. The RHS must therefore be positive for some u.

- The *five* surviving vortex types are
  (i) Standard (Taubes) vortices ($C_0 = -1$, $C = -1$);
  (ii) "Bradlow" vortices ($C_0 = -1$, $C = 0$);
  (iii) Ambjørn–Olesen vortices ($C_0 = -1, C = 1$);
  (iv) Jackiw–Pi vortices ($C_0 = 0$, $C = 1$);
  (v) Popov vortices ($C_0 = 1$, $C = 1$).

- The vortex centres are where e^{2u} vanishes. For standard vortices the magnetic field is maximal there (Meissner effect); for Bradlow vortices the magnetic field is uniform; for the remaining vortex types it is minimal (anti-Meissner effect).
- The existence of solutions, and the moduli space of solutions, are less well understood for the exotic vortices.

Integrable Exotic Vortices

- The vortex equation is integrable on backgrounds with Gaussian curvature $K_0 = C_0$. Integrable backgrounds include
  (i) hyperbolic plane for standard vortices,
  (ii) flat plane or torus for Jackiw–Pi vortices,
  (iii) sphere for Popov vortices.
- In integrable cases, all these vortex equations reduce to Liouville's equation, and solutions are constructed using a holomorphic function $f(z)$.

- The solution is locally

$$|\phi|^2 = e^{2u} = \frac{(1 + C_0|z|^2)^2}{(1 + C|f(z)|^2)^2} \left|\frac{df}{dz}\right|^2 ,$$

and one may fix the gauge by choosing

$$\phi = \frac{1 + C_0|z|^2}{1 + C|f(z)|^2} \frac{df}{dz} .$$

- For example, for planar Jackiw–Pi vortices,

$$\phi = \frac{1}{1 + |f(z)|^2} \frac{df}{dz}$$

with f a rational function. For Jackiw–Pi vortices on a torus f is an elliptic function.

- Vortex centres are the ramification points, where $\frac{df}{dz} = 0$.
- Globally, f is a map from M, with curvature C_0, to a smooth surface with curvature C. $|\phi|^2$ is the ratio of the target metric pulled back to M, and the background metric of M, at corresponding points.
- The pulled-back metric is the Baptista metric and has conical singularities at the ramification points of f.

More Exotic Vortices

- For Popov vortices on a sphere, f is a rational function of degree n. $\frac{df}{dz}$ then has $N = 2n - 2$ zeros, so the vortex number is even.
- There is a (coincident) $N = 2$ Bradlow vortex on the Bolza surface. The Baptista metric is flat, and has one conical singularity with cone angle 6π. This is the metric of a flat regular octagon with opposite sides identified.
- One can find an $N = 6$ Ambjørn–Olesen vortex on the Bolza surface. The vortices are at the branch points of the double covering of the sphere, and the Baptista metric is the pulled-back round metric on the double covered sphere.
- There are many more vortex solutions related to branched covering maps.

Energy and Dynamics of Exotic Vortices

- The static energy function for all the vortex types we have considered is

$$
\begin{aligned}
E &= \int_M \left\{ \frac{1}{\Omega_0^2} f_{12}^2 - \frac{2C}{\Omega_0} \left(\overline{D_1\phi} D_1\phi + \overline{D_2\phi} D_2\phi \right) \right. \\
&\qquad \left. + \left(-C_0 + C|\phi|^2 \right)^2 \right\} \Omega_0 \, d^2x \, .
\end{aligned}
$$

  E is not positive definite for $C > 0$, so not all vortex types are stable.
- Manipulation of E (completing the square) shows that vortices are always stationary points of E, but not always minima.

- The static energy can be extended to a Lagrangian for fields on $\mathbb{R} \times M$, with metric $dt^2 - \Omega_0 \, dz d\bar{z}$,

$$L = \int_M \Big\{ -\frac{1}{2} f_{\mu\nu} f^{\mu\nu} - 2C\overline{D_\mu \phi} D^\mu \phi - \big(-C_0 + C|\phi|^2\big)^2 \Big\} \Omega_0 \, d^2x \, .$$

- The kinetic energy (contribution from terms with $\mu = 0$) is exotic if $C > 0$, as $D_0\phi$ contributes with a minus sign, although the contribution of the electric field f_{0i} is always positive.
- This Lagrangian naturally arises by dimensionally reducing a pure Yang–Mills theory in $4+1$ dimensions (F. Contatto and M. Dunajski). For $C = 1$ and $C = 0$ the gauge group in 4-d is non-compact (SU(1,1) and E_2, resp.) and for $C = 1$ this leads to an exotic kinetic energy.

- Contatto and Dunajski, and also E. Walton and I, are considering the moduli space dynamics of these various vortices. The Strachan/Samols argument again shows that the kinetic energy integral reduces to a sum of localized contributions.
- The kinetic energy and moduli space metric may be positive, zero or negative. Geodesic motion could therefore be along null curves.
- E.g., a Popov vortex at $Z = 0$ is described by the rational function $f(z; t) = c(t)z^2$. The fields, and hence the vortex size, vary as c varies, but the kinetic energy is zero. This follows from the localization formula, and has also been checked by direct integration.
- More generally, for $C = 1$ it is interesting to look at the orbits of the Möbius group of S^2 target transformations, which modify f but leave vortex centres fixed. These fibres of moduli space are non-compact.
- A Jackiw-Pi vortex moving linearly on a torus has zero kinetic energy.

IV. Summary

- The standard Bogomolny/Taubes U(1) vortex equation can be extended to five distinct vortex equations, with parameters C_0 and C. All can be interpreted in terms of the curvature of the Baptista metric $|\phi|^2 ds_0^2$. Vortices are conical singularities of the Baptista metric with cone angle 4π.
- Each vortex equation is integrable on a surface of constant curvature $K_0 = C_0$, and reduces to Liouville's equation. Vortex solutions can be found using holomorphic maps $f(z)$. The Baptista metric has constant curvature $K = C$.
- The metric on the moduli space of vortices is quite well understood using the Strachan/Samols localization formula. For integrable vortices, what is the relation to the Weil–Petersson metric on the moduli space of constant curvature surfaces with conical singularities?

Selected references

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